

## LIFTING AND RESTRICTING RECOLLEMENT DATA

PEDRO NICOLÁS AND MANUEL SAORÍN

ABSTRACT. We study the problem of lifting and restricting TTF triples (equivalently, recollement data) for a certain wide type of triangulated categories. This, together with the parametrizations of TTF triples given in [23], allows us to show that many well-known recollements of right bounded derived categories of algebras are restrictions of recollements in the unbounded level, and leads to criteria to detect recollements of general right bounded derived categories. In particular, we give in Theorem 1 necessary and sufficient conditions for a *right bounded* derived category of a differential graded(=dg) category to be a recollement of right bounded derived categories of dg categories. In Theorem 2 we consider the particular case in which those dg categories are just ordinary algebras.

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*Date:* March 31, 2008.

*1991 Mathematics Subject Classification.* 18E30, 18E40.

*Key words and phrases.* derived category, dg category, recollement.

The authors have been partially supported by research projects from the D.G.I. of the Spanish Ministry of Education and the Fundación Séneca of Murcia, with a part of FEDER funds. The first author has been also supported by the MEC grant AP2003-2896.

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## 1. INTRODUCTION

**1.1. Motivations.** *Torsion torsion-free* ( $=TTF$ ) *triples* are important in the theory of abelian categories (in particular, categories of modules), *cf.* for instance [28]. It turns out that TTF triples still ‘make sense’ in the theory of triangulated categories and that they are also important for they are in bijection with *recollement data* (*cf.* subsection 2.2) and, in many cases, with *smashing subcategories* (*cf.* [24, Proposition 4.4.14], [23, Corollary 2.4]).

Once the problem of parametrizing TTF triples on perfectly generated triangulated categories (in particular, unbounded derived categories of small dg categories) has been essentially solved in [23], we study here the problem of lifting and restricting TTF triples for certain natural full triangulated subcategories which generalize the subcategory of the derived category of an algebra formed by the complexes with right bounded cohomology. A byproduct of our results is an ‘unbounded’ approach to S. König’s work [16].

**1.2. Outline of the paper.** In section 2, we fix some terminology and recall some results on triangulated categories. Also, we introduce the *right bounded derived category* of a small dg category. In section 3, we study the problem of *lifting* a TTF triple from a certain full triangulated subcategory  $\mathcal{D}'$  of a triangulated category  $\mathcal{D}$  with small coproducts and a set of generators contained in  $\mathcal{D}'$ . In subsection 3.1, we consider the general case, and in subsection 3.2 we focus on the case in which  $\mathcal{D}'$  is a kind of ‘right bounded’ triangulated subcategory of  $\mathcal{D}$ . In section 4, we study the problem of *restricting* TTF triples. The general criterion (*cf.* subsection 4.1) was already given by A. A. Beilinson, J. Bernstein and P. Deligne in their seminal paper [3]. In subsection 4.2, we deduce the criterion for the case of a ‘right bounded’ triangulated subcategory. This allows us to regard, in Example 3, some well-known recollements of right bounded derived categories of algebras as restrictions of a recollement induced at the unbounded level by a *homological epimorphism* of the form  $A \rightarrow A/I$  where  $I$  is a two-sided ideal of the algebra  $A$ . With the help of the former sections, we study in section 5 the problem of giving necessary and sufficient conditions for a right bounded derived category of a dg category to be a recollement of right bounded derived categories of dg categories. This leads us to inspect in subsection 5.1 some ‘boundness’ conditions for sets of objects of a right bounded derived category of a dg category. In subsection 5.2, we first give a general criterion (*cf.* Theorem 1) and then a criterion (*cf.* Corollary 5) for the case when the ‘glued’ dg categories have cohomology concentrated in non-positive degrees. This allows us to deduce, in subsection 5.3, a set of necessary and sufficient conditions for the right bounded derived category of an ordinary algebra to be a recollement of right bounded derived categories of ordinary algebras. A result in that direction already appeared in S. König’s paper [16, Theorem 1], but we show in section 6 that stronger assumptions are needed in order S. König’s theorem to be true in general.

## 2. NOTATION AND PRELIMINARY RESULTS

**2.1. Notation.** Unless otherwise stated,  $k$  will be a commutative (associative, unital) ring and every additive category will be assumed to be  $k$ -linear. We will only work with unital algebras and unital modules. We denote by  $\text{Mod } k$  the category of  $k$ -modules. Given a class  $\mathcal{Q}$  of objects of an additive category  $\mathcal{D}$ , we denote by  $\mathcal{Q}^{\perp_{\mathcal{D}}}$ , or  $\mathcal{Q}^{\perp}$  if the category  $\mathcal{D}$  is clear, the full subcategory of  $\mathcal{D}$  formed by the objects  $M$  which are *right orthogonal* to every object of  $\mathcal{Q}$ , *i.e.* such that  $\mathcal{D}(Q, M) = 0$  for all  $Q$  in  $\mathcal{Q}$ . Dually for  ${}^{\perp_{\mathcal{D}}} \mathcal{Q}$ . When  $\mathcal{D}$  is a triangulated category, the *shift functor* will be denoted by  $?[1]$ , and its quasi-inverse will be denoted by  $?[-1]$ . When we use expression like “all the shifts” or “closed under shifts” and so on, we will mean “shifts in both directions”, that is to say, we will refer to the  $n$ th power  $?[n]$  of  $?[1]$  for all the integers  $n \in \mathbf{Z}$ . In case we want to consider another situation (*e.g.* non-negative shifts  $?[n]$ ,  $n \geq 0$ ) this will be said explicitly. If  $\mathcal{Q}$  is a class of objects of a triangulated category  $\mathcal{D}$ :

- (1)  $\mathcal{Q}^+$  will be the class of all non-negative shifts of objects of  $\mathcal{Q}$ .
- (2)  $\text{Sum}_{\mathcal{D}}(\mathcal{Q})$ , or  $\text{Sum}(\mathcal{Q})$  if  $\mathcal{D}$  is clear, will be the class of all small coproducts of objects of  $\mathcal{Q}$ .
- (3)  $\text{aisle}_{\mathcal{D}}(\mathcal{Q})$ , or  $\text{aisle}(\mathcal{Q})$  if  $\mathcal{D}$  is clear, will be the smallest *aisle* (*cf.* [15, Definition 1.1]) in  $\mathcal{D}$  containing  $\mathcal{Q}$ . Notice that  $\text{aisle}_{\mathcal{D}}(\mathcal{Q})$  might not exist since the intersection of aisles might not be an aisle, but if it does then it is closed under small coproducts.
- (4)  $\text{Susp}_{\mathcal{D}}(\mathcal{Q})$ , or  $\text{Susp}(\mathcal{Q})$  if  $\mathcal{D}$  is clear, will be the smallest full *suspended* subcategory (*cf.* [14, subsection 1.1]) of  $\mathcal{D}$  containing  $\mathcal{Q}$  and closed under small coproducts.
- (5)  $\text{Tria}_{\mathcal{D}}(\mathcal{Q})$ , or  $\text{Tria}(\mathcal{Q})$  if  $\mathcal{D}$  is clear, will be the smallest full triangulated subcategory of  $\mathcal{D}$  containing  $\mathcal{Q}$  and closed under small coproducts.

If  $\mathcal{U}$  and  $\mathcal{V}$  are two classes of objects of a triangulated category  $\mathcal{D}$ , then  $\mathcal{U} * \mathcal{V}$  is the class of *extensions* of objects of  $\mathcal{V}$  by objects of  $\mathcal{U}$ , *i.e.* the class formed by those objects  $M$  occurring in a triangle

$$U \rightarrow M \rightarrow V \rightarrow U[1]$$

of  $\mathcal{D}$  with  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ . Notice that the operation  $*$  is associative. For each natural number  $n \geq 0$  the objects of  $\mathcal{U}^{*n} := \mathcal{U} * \dots * \mathcal{U}$  ( $n$  times) are called  *$n$ -fold extensions of length  $n$*  of objects of  $\mathcal{U}$ . We will use without explicit mention the bijection between t-structures on a triangulated category  $\mathcal{D}$  and aisles in  $\mathcal{D}$ , proved by B. Keller and D. Vossieck in [15]. If  $(\mathcal{U}, \mathcal{V}[1])$  is a t-structure on a triangulated category  $\mathcal{D}$ , we denote by  $u : \mathcal{U} \hookrightarrow \mathcal{D}$  and  $v : \mathcal{V} \hookrightarrow \mathcal{D}$  the inclusion functors, by  $\tau_{\mathcal{U}}$  a right adjoint to  $u$  and by  $\tau^{\mathcal{V}}$  a left adjoint to  $v$ .

**2.2. TTF triples and recollement data.** A *torsion torsionfree* ( $=$  *TTF*) *triple* on a triangulated category  $\mathcal{D}$  is a triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  of full subcategories of  $\mathcal{D}$  such that  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{Y}, \mathcal{Z})$  are t-structures on  $\mathcal{D}$ . Notice that, in particular,  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are full triangulated subcategories of  $\mathcal{D}$ . It is well known that TTF triples are in bijection with (suitable equivalence classes of) *recollement data* (*cf.* [3, 1.4.4], [21, subsection 9.2], [24, subsection 4.2]). For the convenience of the reader we recall

how this bijection works. If

$$\begin{array}{ccccc} & i^* & & j^* & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{D}_F & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j_*} & \mathcal{D}_U \\ & \curvearrowleft & & \curvearrowleft & \\ & i^! & & j^! & \end{array}$$

expresses  $\mathcal{D}$  as a recollement of  $\mathcal{D}_F$  and  $\mathcal{D}_U$ , then

$$(j_!(\mathcal{D}_U), i_*(\mathcal{D}_F), j_*(\mathcal{D}_U))$$

is a TTF triple on  $\mathcal{D}$ , where by  $j_!(\mathcal{D}_U)$  we mean the essential image of  $j_!$ , and analogously with the other functors. Conversely, if  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  is a TTF triple on  $\mathcal{D}$ , then  $\mathcal{D}$  is a recollement of  $\mathcal{Y}$  and  $\mathcal{X}$  as follows:

$$\begin{array}{ccccc} & \tau^{\mathcal{Y}} & & z\tau^{\mathcal{Z}}x & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{Y} & \xrightarrow{y} & \mathcal{D} & \xrightarrow{\tau_{\mathcal{X}}} & \mathcal{X} \\ & \curvearrowleft & & \curvearrowleft & \\ & \tau_{\mathcal{Y}} & & x & \end{array}$$

Notice that for a TTF triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  the compositions  $\mathcal{X} \xrightarrow{x} \mathcal{D} \xrightarrow{\tau^{\mathcal{Z}}} \mathcal{Z}$  and  $\mathcal{Z} \xrightarrow{z} \mathcal{D} \xrightarrow{\tau^{\mathcal{X}}} \mathcal{X}$  are mutually quasi-inverse triangle equivalences (cf. [24, Lemma 1.6.7]).

**2.3. (Super)perfectness and compactness.** An object  $P$  of a triangulated category  $\mathcal{D}$  is *perfect* (respectively, *superperfect*) if for every countable (respectively, small) family of morphisms  $M_i \rightarrow N_i$ ,  $i \in I$ , of  $\mathcal{D}$  such that the coproducts  $\coprod_I M_i$  and  $\coprod_I N_i$  exist, the induced map

$$\mathcal{D}(P, \coprod_I M_i) \rightarrow \mathcal{D}(P, \coprod_I N_i)$$

is surjective provided every map

$$\mathcal{D}(P, M_i) \rightarrow \mathcal{D}(P, N_i), \quad i \in I$$

is surjective. Particular cases of superperfect objects are *compact* objects, *i.e.* objects  $P$  such that the functor  $\mathcal{D}(P, ?)$  preserves small coproducts.

**2.4. Milnor colimits.** Now we recall a crucial construction which formally imitates the construction of the direct limit in an abelian category. Let  $\mathcal{D}$  be a triangulated category and let

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$$

be a sequence of morphisms of  $\mathcal{D}$  such that the coproduct  $\coprod_{n \geq 0} M_n$  exists in  $\mathcal{D}$ . The *Milnor colimit* of this sequence, denoted by  $\text{Mcolim } M_n$ , is given, up to non-unique isomorphism, by the triangle

$$\coprod_{n \geq 0} M_n \xrightarrow{1-\sigma} \coprod_{n \geq 0} M_n \xrightarrow{\pi} \text{Mcolim } M_n \rightarrow \coprod_{n \geq 0} M_n[1],$$

where the morphism  $\sigma$  has components

$$M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{\text{can}} \coprod_{p \geq 0} M_p.$$

The above triangle is the *Milnor triangle* (cf. [20, 12]) associated to the sequence  $f_n$ ,  $n \geq 0$ . The notion of Milnor colimit has appeared in the literature under the name of *homotopy colimit* (cf. [4, Definition 2.1], [21, Definition 1.6.4]) and

*homotopy limit* (cf. [11, subsection 5.1]). However, we think it is better to keep this terminology for the notions appearing in the theory of derivators [18, 19, 5] and in the theory of model categories [9].

**2.5. Generation of triangulated categories.** Let us consider three ways in which a triangulated category  $\mathcal{D}$  can be generated by a class  $\mathcal{Q}$  of objects:

- 1)  $\mathcal{D}$  is *generated* by  $\mathcal{Q}$  if an object  $M$  of  $\mathcal{D}$  is zero whenever

$$\mathcal{D}(Q[n], M) = 0$$

for every object  $Q$  of  $\mathcal{Q}$  and every integer  $n \in \mathbf{Z}$ . In this case, we say that  $\mathcal{Q}$  is a *class of generators* of  $\mathcal{D}$  and that  $\mathcal{Q}$  *generates*  $\mathcal{D}$ . A triangulated category with small coproducts is *compactly generated* if it is generated by a set of compact objects.

- 2)  $\mathcal{D}$  satisfies the *principle of infinite dévissage* with respect to  $\mathcal{Q}$  if  $\mathcal{D} = \text{Tria}_{\mathcal{D}}(\mathcal{Q})$ . In this situation,  $\mathcal{Q}$  generates  $\mathcal{D}$ .

- 3)  $\mathcal{D}$  is *exhaustively generated* by  $\mathcal{Q}$  if the following conditions hold:

3.1) Small coproducts of objects of  $\bigcup_{m \geq 0} \text{Sum}(\mathcal{Q})^{*m}$  exist in  $\mathcal{D}$ .

3.2) For each object  $M$  of  $\mathcal{D}$  there exists an integer  $i \in \mathbf{Z}$  and a triangle

$$\coprod_{n \geq 0} Q_n \rightarrow \coprod_{n \geq 0} Q_n \rightarrow M[i] \rightarrow \coprod_{n \geq 0} Q_n[1]$$

in  $\mathcal{D}$  with  $Q_n \in \bigcup_{m \geq 0} \text{Sum}(\mathcal{Q})^{*m}$ .

Notice that, in this situation,  $\mathcal{D}$  satisfies the principle of infinite dévissage with respect to  $\mathcal{Q}$ . If  $\mathcal{Q} = \mathcal{P}^+$  for some set  $\mathcal{P}$ , then we also say that  $\mathcal{D}$  is *exhaustively generated to the left* by  $\mathcal{P}$ .

The following are two examples of exhaustively generated triangulated categories:

**Example 1.** Let  $\mathcal{D}$  be a triangulated category with small coproducts, and let  $\mathcal{P}$  be a set of objects of  $\mathcal{D}$  which are perfect in  $\text{Tria}(\mathcal{P})$ . As proved by [17, Theorem A], every object of  $\text{Tria}(\mathcal{P})$  is the Milnor colimit of a sequence

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \rightarrow \dots$$

of morphisms of  $\mathcal{D}$  where  $P_n$  is an  $n$ th extension of small coproducts of shifts of objects of  $\mathcal{P}$ . This shows that  $\text{Tria}(\mathcal{P})$  is exhaustively generated by the set formed by all the shifts of objects of  $\mathcal{P}$ . In particular, the derived category  $\mathcal{DA}$  of a small dg category  $\mathcal{A}$  is exhaustively generated by all the shifts of the representable modules  $A^\wedge := \mathcal{A}(\cdot, A)$ ,  $A \in \mathcal{A}$ .

**Example 2.** Let  $\mathcal{D}$  be a triangulated category with small coproducts, and let  $\mathcal{P}$  be a set of perfect objects of  $\mathcal{D}$ . As proved in [27, Theorem 2.2], we have that  $\text{Susp}(\mathcal{P})$  is an aisle in  $\mathcal{D}$  and every object of  $\text{Susp}(\mathcal{P})$  is a Milnor colimit of a sequence

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \rightarrow \dots$$

of morphisms of  $\mathcal{D}$  where  $P_n$  is an  $n$ th extension of small coproduct of non-negative shifts of objects of  $\mathcal{P}$ . In particular,  $\bigcup_{n \in \mathbf{Z}} \text{Susp}(\mathcal{P})[n]$  is exhaustively generated to the left by  $\mathcal{P}$ .

**2.6. The right bounded derived category of a dg category.** Let  $\mathcal{A}$  be a small dg category. Since the representable dg right  $\mathcal{A}$ -modules  $A^\wedge$ ,  $A \in \mathcal{A}$ , are compact objects of the derived category  $\mathcal{DA}$  of  $\mathcal{A}$ , then  $\text{Susp}(\{A^\wedge\}_{A \in \mathcal{A}})$  is an aisle in  $\mathcal{DA}$ , which will be denoted by  $\mathcal{D}^{\leq 0}\mathcal{A}$ . Its associated coaisle, denoted by  $\mathcal{D}^{> 0}\mathcal{A}$ , consists of those modules  $M$  with cohomology concentrated in positive degrees, *i.e.*  $H^n M(A) = 0$  for each  $A \in \mathcal{A}$  and  $n \leq 0$ . For each integer  $n \in \mathbf{Z}$  we put

$$\mathcal{D}^{\leq n}\mathcal{A} := \mathcal{D}^{\leq 0}\mathcal{A}[-n]$$

and

$$\mathcal{D}^{> n}\mathcal{A} := \mathcal{D}^{> 0}\mathcal{A}[-n],$$

and denote by  $\tau^{\leq n}$  and  $\tau^{> n}$  the torsion and torsionfree functors, respectively, corresponding to the t-structure  $(\mathcal{D}^{\leq n}\mathcal{A}, \mathcal{D}^{> n}\mathcal{A})$ . The following lemma ensures that, in case the dg category  $\mathcal{A}$  has cohomology concentrated in non-positive degrees, the aisle  $\mathcal{D}^{\leq n}\mathcal{A}$  admits a familiar description in terms of cohomology.

**Lemma 1.** *Let  $\mathcal{A}$  be a small dg category with cohomology concentrated in degrees  $(-\infty, m]$  for some integer  $m \in \mathbf{Z}$ . For a dg  $\mathcal{A}$ -module  $M$  we consider the following assertions:*

- 1)  $M \in \mathcal{D}^{\leq s}\mathcal{A}$ .
- 2)  $H^i M(A) = 0$  for each integer  $i > m + s$  and every object  $A$  of  $\mathcal{A}$ .

Then 1)  $\Rightarrow$  2) and, in case  $m = 0$ , we also have 2)  $\Rightarrow$  1).

*Proof.* 1)  $\Rightarrow$  2) Since  $M[s]$  belongs to  $\text{Susp}(\{A^\wedge\}_{A \in \mathcal{A}})$ , there exists a triangle in  $\mathcal{DA}$

$$\coprod_{n \geq 0} P_n \rightarrow \coprod_{n \geq 0} P_n \rightarrow M[s] \rightarrow \coprod_{n \geq 0} P_n[1]$$

with  $P_n \in \text{Sum}(\{A^\wedge\}_{A \in \mathcal{A}}^+)^{*n}$  for each  $n \geq 0$  (cf. for instance Example 2). Then, for each  $A \in \mathcal{A}$  we get the long exact sequence of cohomology

$$\dots \rightarrow \coprod_{n \geq 0} H^i P_n(A) \rightarrow H^{i+s} M(A) \rightarrow \coprod_{n \geq 0} H^{i+1} P_n(A) \rightarrow \dots$$

with  $H^i P_n(A) \cong (\mathcal{DA})(A^\wedge, P_n[i]) = 0$  for each  $i > m$ .

2)  $\Rightarrow$  1) Consider the triangle in  $\mathcal{DA}$

$$M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$$

with  $M' \in \mathcal{D}^{\leq s}\mathcal{A}$  and  $M'' \in (\mathcal{D}^{\leq s}\mathcal{A})^\perp$ . In particular,  $H^i M''(A) = 0$  for each  $A \in \mathcal{A}$  and each  $i \leq s$ . The aim is to prove that  $H^i M''(A) = 0$  for each  $A \in \mathcal{A}$  and each  $i \in \mathbf{Z}$ . Thus, consider the long exact sequence of cohomology

$$\dots \rightarrow H^i M(A) \rightarrow H^i M''(A) \rightarrow H^{i+1} M'(A) \rightarrow \dots$$

By using 1) and the extra assumption on  $\mathcal{A}$ , we have that  $H^i M'(A) = 0$  for each  $i > s$  and, by hypothesis,  $H^i M(A) = 0$  for each  $i > s$ . This implies that  $H^i M''(A) = 0$  for each  $i > s$ .  $\checkmark$

For an arbitrary small dg category  $\mathcal{A}$ , the t-structure  $(\mathcal{D}^{\leq 0}\mathcal{A}, \mathcal{D}^{> 0}\mathcal{A})$  is said to be the *canonical t-structure* on  $\mathcal{DA}$ . We will write

$$\mathcal{D}^-\mathcal{A} := \bigcup_{n \in \mathbf{Z}} \mathcal{D}^{\leq n}\mathcal{A},$$

and we will refer to  $\mathcal{D}^-\mathcal{A}$  as the *right bounded derived category* of  $\mathcal{A}$ . These names are justified by the Lemma 1.

**Remark 1.** Notice that  $\mathcal{D}^-\mathcal{A}$  is not closed under small coproducts in  $\mathcal{DA}$ . Indeed, given  $A \in \mathcal{A}$ , the coproduct  $\coprod_{n \in \mathbb{Z}} A^\wedge[n]$  does not belong to  $\mathcal{D}^-\mathcal{A}$ . Also, notice that  $\mathcal{D}^-\mathcal{A}$  is exhaustively generated to the left by the free  $\mathcal{A}$ -modules  $A^\wedge$ ,  $A \in \mathcal{A}$ .

### 3. LIFTING OF TTF TRIPLES

#### 3.1. General criterion.

**Definition 1.** Let  $\mathcal{D}$  be a triangulated category and let  $\mathcal{D}'$  be a full triangulated subcategory of  $\mathcal{D}$ . We say that a TTF triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  on  $\mathcal{D}$  *restricts to* or *is a lifting of* a TTF triple  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  on  $\mathcal{D}'$  if we have

$$(\mathcal{X} \cap \mathcal{D}', \mathcal{Y} \cap \mathcal{D}', \mathcal{Z} \cap \mathcal{D}') = (\mathcal{X}', \mathcal{Y}', \mathcal{Z}').$$

That is to say,  $\mathcal{X}'$  is the full subcategory of  $\mathcal{D}'$  formed by those objects of  $\mathcal{D}'$  which are in  $\mathcal{X}$ , and analogously with the other subcategories. In this case, we say that  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  *lifts to* or *is the restriction of*  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ .

**Definition 2.** A class  $\mathcal{P}$  of objects of a triangulated category  $\mathcal{D}$  is *recollement-defining* if the class  $\mathcal{Y}$  of those objects which are right orthogonal to all the shifts of objects of  $\mathcal{P}$  is both an aisle and a coaisle in  $\mathcal{D}$ , i.e.  $\mathcal{Y}$  fits in a TTF triple  $({}^\perp\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^\perp)$  on  $\mathcal{D}$ .

**Proposition 1.** Let  $\mathcal{D}$  be a triangulated category with small coproducts and let  $\mathcal{D}'$  be a full triangulated subcategory containing a set  $\mathcal{Q}$  of generators of  $\mathcal{D}$ . For a TTF triple  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  on  $\mathcal{D}'$  the following assertions are equivalent:

- 1)  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  is the restriction of a TTF triple on  $\mathcal{D}$ .
- 2) There is a set  $\mathcal{P}$  of objects of  $\mathcal{X}'$  such that:
  - 2.1)  $\mathcal{P}$  is recollement-defining in  $\mathcal{D}$ .
  - 2.2) If an object of  $\mathcal{D}$  is right orthogonal to all the shifts of objects of  $\mathcal{P}$ , then it is right orthogonal to all the objects of  $\mathcal{X}'$ .
- 3) The objects of  $\mathcal{X}'$  form a recollement-defining class of  $\mathcal{D}$ .

Moreover, we can take  $\mathcal{P} = (\tau_{\mathcal{X}'} z' \tau^{\mathcal{Z}'})(\mathcal{Q})$ .

*Proof.* 1)  $\Rightarrow$  2) Let  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  be a TTF triple on  $\mathcal{D}$  which restricts to  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$ , and let  $\mathcal{Q}$  be a set of generators of  $\mathcal{D}$  contained in  $\mathcal{D}'$ . Notice that, for each object  $Q$  of  $\mathcal{Q}$ , the torsion triangle associated to the t-structure  $(\mathcal{Y}, \mathcal{Z})$  can be taken to be

$$\tau_{\mathcal{Y}'}(Q) \rightarrow Q \rightarrow \tau^{\mathcal{Z}'}(Q) \rightarrow \tau_{\mathcal{Y}'}(Q)[1].$$

Then, it is straightforward to check that  $\tau^{\mathcal{Z}'}(Q)$  is a set of generators of  $\mathcal{Z}$ . Since the composition

$$\mathcal{Z} \xrightarrow{z} \mathcal{D} \xrightarrow{\tau_{\mathcal{X}}} \mathcal{X}$$

is a triangle equivalence, we have that  $\mathcal{P} := (\tau_{\mathcal{X}} z \tau^{\mathcal{Z}})(\mathcal{Q})$  is a set of generators of  $\mathcal{X}$ . But, since  $\tau^{\mathcal{Z}'}(Q)$  is contained in  $\mathcal{D}'$ , then we have  $\mathcal{P} = (\tau_{\mathcal{X}'} z' \tau^{\mathcal{Z}'})(\mathcal{Q})$ , which is contained in  $\mathcal{X}'$ . The fact that  $(\mathcal{X}, \mathcal{Y})$  is a t-structure on  $\mathcal{D}$  implies that  $\mathcal{Y}$  is the set of objects of  $\mathcal{D}$  which are right orthogonal to all the shifts of objects of  $\mathcal{P}$ , and so  $\mathcal{P}$  is recollement-defining in  $\mathcal{D}$ . Finally, the inclusions  $\mathcal{Y} \subseteq \mathcal{X}^\perp \subseteq \mathcal{X}'^\perp$  prove 2.2). 2)  $\Rightarrow$  3) is clear.

3)  $\Rightarrow$  1) Consider  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) := ({}^\perp(\mathcal{X}'^\perp), \mathcal{X}'^\perp, (\mathcal{X}'^\perp)^\perp)$ , with orthogonals taken in  $\mathcal{D}$ , which is a TTF triple on  $\mathcal{D}$ . Since  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  is a TTF triple on  $\mathcal{D}'$ , then

we have  $\mathcal{Y}' = \mathcal{X}'^\perp \cap \mathcal{D}' = \mathcal{Y} \cap \mathcal{D}'$ . Let us prove now  $\mathcal{X}' = \mathcal{X} \cap \mathcal{D}'$ . The inclusion  $\subseteq$  is clear. Conversely, let  $X$  be an object of  $\mathcal{X} \cap \mathcal{D}'$  and consider the triangle

$$\tau_{\mathcal{X}'}(X) \rightarrow X \rightarrow \tau^{\mathcal{Y}'}(X) \rightarrow \tau_{\mathcal{X}'}(X)[1].$$

Its two terms on the left belong to  $\mathcal{X}$ . Then  $\tau^{\mathcal{Y}'}(X) \in \mathcal{X} \cap \mathcal{Y}' \subseteq \mathcal{X} \cap \mathcal{Y} = \{0\}$  and so  $X \in \mathcal{X}'$ . Now, we have the following inclusions

$$\mathcal{Z} \cap \mathcal{D}' = \mathcal{Y}^\perp \cap \mathcal{D}' \subseteq \mathcal{Y}'^\perp \cap \mathcal{D}' = \mathcal{Z}'.$$

Finally, let  $\mathcal{Q}$  be the set of generators of  $\mathcal{D}$  contained in  $\mathcal{D}'$ . It is easy to prove that  $\tau^{\mathcal{Y}}(\mathcal{Q})^\perp = \mathcal{Z}$ . Also, notice that  $\tau^{\mathcal{Y}}(\mathcal{Q}) \subseteq \mathcal{Y} \cap \mathcal{D}' = \mathcal{Y}'$ . Therefore,

$$\mathcal{Z}' = \mathcal{Y}'^\perp \cap \mathcal{D}' \subseteq \tau^{\mathcal{Y}}(\mathcal{Q})^\perp \cap \mathcal{D}' = \mathcal{Z} \cap \mathcal{D}'.$$

✓

**Corollary 1.** *Under the hypotheses of Proposition 1, the map*

$$(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \mapsto (\mathcal{X} \cap \mathcal{D}', \mathcal{Y} \cap \mathcal{D}', \mathcal{Z} \cap \mathcal{D}')$$

*defines a bijection between:*

- 1) *TTF triples on  $\mathcal{D}$  which restricts to TTF triples on  $\mathcal{D}'$ .*
- 2) *TTF triples on  $\mathcal{D}'$  which are restriction of TTF triples on  $\mathcal{D}$ .*

*Proof.* Of course, the map is surjective. Now, let  $\mathcal{Q} \subseteq \mathcal{D}'$  a set of generators of  $\mathcal{D}$  and let  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  be a TTF triple such that  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}') = (\mathcal{X} \cap \mathcal{D}', \mathcal{Y} \cap \mathcal{D}', \mathcal{Z} \cap \mathcal{D}')$  is a TTF triple on  $\mathcal{D}'$ . Then, the proof of Proposition 1 shows that  $\mathcal{Y}$  is precisely the class of objects of  $\mathcal{D}$  which are right orthogonal to all the shifts of objects of  $(\tau_{\mathcal{X}'} \tau^{\mathcal{Z}'})(\mathcal{Q})$ . This implies the injectivity. ✓

**3.2. ‘Right bounded’ triangulated subcategories.** Let  $\mathcal{Q}$  be a set of objects of a triangulated category  $\mathcal{D}$  with small coproducts. Let us assume that  $\text{Susp}(\mathcal{Q})$  is an aisle in  $\mathcal{D}$ . This is the case, for instance if the objects of  $\mathcal{Q}$  are perfect (cf. [17, 27]). Notice that, in case  $\text{Susp}(\mathcal{Q})$  is an aisle in  $\mathcal{D}$ , then  $\text{Susp}(\mathcal{Q}) = \text{aisle}(\mathcal{Q})$ , i.e.  $\text{Susp}(\mathcal{Q})$  is the smallest aisle in  $\mathcal{D}$  containing  $\mathcal{Q}$ . We are interested in the interplay between TTF triples on abstract ‘unbounded’ triangulated categories and TTF triples on abstract ‘right bounded triangulated’ categories. More precisely, we are interested in the interplay between TTF triples on  $\mathcal{D}$  and TTF triples on the full triangulated subcategory  $\mathcal{D}' := \bigcup_{n \in \mathbb{Z}} \text{aisle}(\mathcal{Q})[n]$  of  $\mathcal{D}$ . A good example to keep in mind is  $\mathcal{D} = \mathcal{DA}$  and  $\mathcal{D}' = \mathcal{D}^- \mathcal{A}$  for a small dg category  $\mathcal{A}$ . First we need to understand better the interplay between  $\mathcal{D}$  and  $\mathcal{D}'$ .

**Lemma 2.** *The following assertions hold:*

- 1) *The inclusion functor  $\iota : \mathcal{D}' \hookrightarrow \mathcal{D}$  preserves small coproducts.*
- 2) *If  $\text{Susp}(\mathcal{Q})^\perp$  is closed under small coproducts, then an object  $P$  of  $\mathcal{D}'$  is compact (respectively, perfect, superperfect) in  $\mathcal{D}'$  if and only if it is compact (respectively, perfect, superperfect) in  $\mathcal{D}$ .*

*Proof.* 1) Let  $D'_i$ ,  $i \in I$ , be a family of objects of  $\mathcal{D}'$  whose coproduct exists in  $\mathcal{D}'$ . We write  $\coprod_{i \in I} D'_i$  for the coproduct in  $\mathcal{D}$ ,  $D'$  for the coproduct in  $\mathcal{D}'$  and  $v_i : D'_i \rightarrow D'$  for the canonical morphisms. For simplicity, put  $\text{aisle}(\mathcal{Q})[k] = \mathcal{U}_k$ . Therefore, we have a chain

$$\cdots \subseteq \mathcal{U}_{k+1} \subseteq \mathcal{U}_k \subseteq \mathcal{U}_{k-1} \subseteq \cdots \subseteq \mathcal{D}'$$

of aisles in  $\mathcal{D}$  whose union is  $\mathcal{D}'$ .



*Claim:* If  $m, n \in \mathbf{Z}$  are integers such that  $D' \in \mathcal{U}_n$  and  $D'_i \in \mathcal{U}_m \setminus \mathcal{U}_{m+1}$  for some  $i \in I$ , then  $n \leq m$ . Indeed, fix such an  $i$  and assume  $n > m$  and consider the triangle

$$\tau_{\mathcal{U}_n}(D'_i) \rightarrow D'_i \xrightarrow{f} \tau^{\mathcal{U}_n^\perp}(D'_i) \rightarrow \tau_{\mathcal{U}_n}(D'_i)[1].$$

Since the two first vertices of this triangle belong to  $\mathcal{D}'$ , then so does  $\tau^{\mathcal{U}_n^\perp}(D'_i)$ . Hence, by using the universal property of the coproduct, we have that  $f$  induces a morphism

$$\tilde{f}: D' \rightarrow \tau^{\mathcal{U}_n^\perp}(D'_i)$$

such that

$$\tilde{f}v_j = \begin{cases} f & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $D' \in \mathcal{U}_n$ , then  $\tilde{f} = 0$  and so  $f = 0$ . Therefore,  $D'_i$  is a direct summand of  $\tau_{\mathcal{U}_n}(D'_i)$ . This implies that  $D'_i$  belongs to  $\mathcal{U}_n$ , and so it belongs to  $\mathcal{U}_{m+1}$ , which is a contradiction. Consider the following two situations:

*First situation:* For each  $i \in I$  we have  $D'_i \in \bigcap_{k \in \mathbf{Z}} \mathcal{U}_k$ . Since aisles are closed under small coproducts, this implies that the coproduct  $\coprod_{i \in I} D'_i$  belongs to  $\bigcap_{k \in \mathbf{Z}} \mathcal{U}_k$ , and so to  $\mathcal{D}'$ . Hence  $D' \cong \coprod_{i \in I} D'_i$ . *Second situation:* There exists  $j \in I$  such that  $D'_j \in \mathcal{U}_m \setminus \mathcal{U}_{m+1}$ . Given  $i \in I$ , put  $m_i$  for the maximum of the set of those integers  $k \in \mathbf{Z}$  such that  $D'_i \in \mathcal{U}_k$ . Put  $m_i = \infty$  if  $D'_i \in \bigcap_{k \in \mathbf{Z}} \mathcal{U}_k$ . Thanks to the claim, we know that, in any case,  $m_i \geq n$  for each  $i \in I$ . Then  $D'_i \in \mathcal{U}_n$  for every  $i \in I$ , and so  $\coprod_{i \in I} D'_i \in \mathcal{U}_n$ . Again, this implies  $\coprod_{i \in I} D'_i \cong D'$ . 2) Assertion 1) implies that if  $P \in \mathcal{D}'$  is compact in  $\mathcal{D}$  then it is also compact in  $\mathcal{D}'$ . Conversely, let  $P \in \mathcal{D}'$  be compact in  $\mathcal{D}'$  and fix an integer  $n \in \mathbf{Z}$  such that  $P \in \mathcal{U}_n$ . If  $D_i, i \in I$ , is a family of objects of  $\mathcal{D}$ , then we have isomorphisms

$$\mathcal{D}(P, D_i) \cong \mathcal{U}_n(P, \tau_{\mathcal{U}_n}(D_i)) = \mathcal{D}'(P, \tau_{\mathcal{U}_n}(D_i))$$

for each  $i \in I$ , and

$$\mathcal{D}(P, \coprod_{i \in I} D_i) \cong \mathcal{U}_n(P, \tau_{\mathcal{U}_n}(\coprod_{i \in I} D_i)) = \mathcal{D}'(P, \tau_{\mathcal{U}_n}(\coprod_{i \in I} D_i)).$$

By hypothesis,  $\mathcal{U}_n^\perp$  is closed under small coproducts. This is equivalent to the fact that  $\tau_{\mathcal{U}_n}$  preserves small coproducts, and so we have a canonical isomorphism

$$\coprod_{i \in I} \tau_{\mathcal{U}_n}(D_i) \xrightarrow{\sim} \tau_{\mathcal{U}_n}(\coprod_{i \in I} D_i).$$

Finally, we have the commutative diagram

$$\begin{array}{ccc} \coprod_{i \in I} \mathcal{D}(P, D_i) & \xrightarrow{\sim} & \coprod_{i \in I} \mathcal{D}'(P, \tau_{\mathcal{U}_n}(D_i)) \\ \downarrow \text{can} & & \downarrow \wr \text{can} \\ & & \mathcal{D}'(P, \coprod_{i \in I} \tau_{\mathcal{U}_n}(D_i)) \\ & & \downarrow \wr \text{can} \\ \mathcal{D}(P, \coprod_{i \in I} D_i) & \xrightarrow{\sim} & \mathcal{D}'(P, \tau_{\mathcal{U}_n}(\coprod_{i \in I} D_i)) \end{array}$$

where the morphisms ‘can’ are the canonical ones. This proves that  $P$  is compact in  $\mathcal{D}$ . The case of  $P$  being (super)perfect follows similarly using adjunction.  $\checkmark$

**Proposition 2.** *Assume that  $\mathcal{Q}$  is a set of perfect generators of  $\mathcal{D}$  such that  $\text{aisle}(\mathcal{Q})^\perp$  is closed under small coproducts. Let  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  be a TTF triple on  $\mathcal{D}'$  such that  $\mathcal{X}'$  is exhaustively generated to the left by a set  $\mathcal{P}$  whose objects are superperfect in  $\mathcal{X}'$ . Then,*

- 1) *The objects of  $\mathcal{P}$  are superperfect in  $\mathcal{D}'$ .*
- 2)  *$(\text{Tri}_{\mathcal{D}}(\mathcal{P}), \text{Tri}_{\mathcal{D}}(\mathcal{P})^\perp, (\text{Tri}_{\mathcal{D}}(\mathcal{P})^\perp)^\perp)$  is a TTF triple on  $\mathcal{D}$  which restricts to  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$ .*

*Proof.* 1) Let  $P$  be an object of  $\mathcal{P}$ . Let  $\alpha_i : M_i \rightarrow N_i$ ,  $i \in I$ , be a family of morphism of  $\mathcal{D}'$  such that the induced maps  $\mathcal{D}'(P, M_i) \rightarrow \mathcal{D}'(P, N_i)$ ,  $i \in I$ , are surjective. In other words, the maps  $\mathcal{X}'(P, \tau_{\mathcal{X}'} M_i) \rightarrow \mathcal{X}'(P, \tau_{\mathcal{X}'} N_i)$ ,  $i \in I$ , are surjective. Assume the coproducts  $\coprod_I M_i$  and  $\coprod_I N_i$  exist in  $\mathcal{D}'$ . We have to prove that the induced map  $\mathcal{D}'(P, \coprod_I M_i) \rightarrow \mathcal{D}'(P, \coprod_I N_i)$  is surjective. For each  $i \in I$  we consider the triangle

$$x' \tau_{\mathcal{X}'} M_i \xrightarrow{f_i} M_i \xrightarrow{g_i} y' \tau^{\mathcal{Y}'} M_i \xrightarrow{h_i} x' \tau_{\mathcal{X}'} M_i[1]$$

of  $\mathcal{D}'$  associated to the t-structure  $(\mathcal{X}', \mathcal{Y}')$ . Since both  $\tau^{\mathcal{Y}'}$  and  $y'$  preserve small coproducts, the coproduct  $\coprod_I y' \tau^{\mathcal{Y}'} M_i$  exists in  $\mathcal{D}'$  and the canonical morphism  $\coprod_I y' \tau^{\mathcal{Y}'} M_i \rightarrow y' \tau^{\mathcal{Y}'} \coprod_I M_i$  is an isomorphism. The existence of the coproducts  $\coprod_I M_i$  and  $\coprod_I y' \tau^{\mathcal{Y}'} M_i$  implies that the coproduct  $\coprod_I x' \tau_{\mathcal{X}'} M_i$  exists in  $\mathcal{D}'$  and that

$$\coprod_I x' \tau_{\mathcal{X}'} M_i \xrightarrow{\coprod_I f_i} \coprod_I M_i \xrightarrow{\coprod_I g_i} \coprod_I y' \tau^{\mathcal{Y}'} M_i \xrightarrow{\coprod_I h_i} \coprod_I x' \tau_{\mathcal{X}'} M_i[1]$$

is a triangle of  $\mathcal{D}'$ . Hence, the canonical morphism  $\coprod_I x' \tau_{\mathcal{X}'} M_i \rightarrow x' \tau_{\mathcal{X}'} \coprod_I M_i$  is an isomorphism. Of course, we can proceed similarly with the objects  $N_i$ ,  $i \in I$ . Then, the map  $\mathcal{D}'(P, \coprod_I M_i) \rightarrow \mathcal{D}'(P, \coprod_I N_i)$  is isomorphic to  $\mathcal{X}'(P, \coprod_I \tau_{\mathcal{X}'} M_i) \rightarrow \mathcal{X}'(P, \coprod_I \tau_{\mathcal{X}'} N_i)$ , which is surjective since  $P$  is superperfect in  $\mathcal{X}'$ . 2) Lemma 2 implies that the objects of  $\mathcal{P}$  are also superperfect in  $\mathcal{D}$ . Then, by using Brown representability theorem [17] we deduce that  $\mathcal{X} := \text{Tri}_{\mathcal{D}}(\mathcal{P})$  is an aisle in  $\mathcal{D}$ . Notice that the corresponding coaisle  $\mathcal{Y} := \text{Tri}_{\mathcal{D}}(\mathcal{P})^\perp$  is formed by those objects which are right orthogonal to all the shifts of objects of  $\mathcal{P}$ , and so it is closed under small coproducts. Since  $\tau^{\mathcal{Y}}(\mathcal{Q})$  is a set of perfect generators of  $\mathcal{Y}$ , we can use again Brown representability theorem to deduce that  $\mathcal{Y}$  is an aisle in  $\mathcal{D}$ . Put  $\mathcal{Z} := \mathcal{Y}^\perp$ . Of course, condition 2.1) of Proposition 1 is satisfied. Let us prove that condition 2.2) of this proposition also holds. For this, first notice that thanks to Lemma 2, we know that the inclusion functor  $\mathcal{X}' \hookrightarrow \mathcal{D}$  preserves small coproducts for it is the composition of the coproduct-preserving inclusions  $\mathcal{X}' \hookrightarrow \mathcal{D}' \hookrightarrow \mathcal{D}$ . Now, for every object  $X \in \mathcal{X}'$  there exists an integer  $i \in \mathbb{Z}$  such that  $X$  fits into a triangle of  $\mathcal{X}'$  (and so of  $\mathcal{D}$ )

$$\coprod_{n \geq 0} P_n[i] \rightarrow \coprod_{n \geq 0} P_n[i] \rightarrow X \rightarrow \coprod_{n \geq 0} P_n[i+1]$$

with  $P_n \in \bigcup_{m \geq 0} \text{Sum}(\mathcal{P}^+)^{*m}$  for each  $n \geq 0$ . Let  $M$  be an object of  $\mathcal{D}$  which is right orthogonal to all the shifts of objects of  $\mathcal{P}$ . Then, we get a long exact sequence

$$\dots \rightarrow \mathcal{D}(\coprod_{n \geq 0} P_n[i+1], M) \rightarrow \mathcal{D}(X, M) \rightarrow \mathcal{D}(\coprod_{n \geq 0} P_n[i], M) \rightarrow \dots$$

in which

$$\mathcal{D}(\coprod_{n \geq 0} P_n[i+1], M) = \mathcal{D}(\coprod_{n \geq 0} P_n[i], M) = 0,$$

and so  $\mathcal{D}(X, M) = 0$ .

✓

#### 4. RESTRICTION OF TTF TRIPLES

**4.1. General criterion.** The general criterion to restrict t-structures is the following well-known lemma, which already appeared in the work of A. A. Beilinson, J. Bernstein and P. Deligne (cf. [3, paragraph 1.3.19]):

**Lemma 3.** *Let  $(\mathcal{U}, \mathcal{V}[1])$  be a t-structure on a triangulated category  $\mathcal{D}$ , and let  $\mathcal{D}'$  be a strictly (=closed under isomorphisms) full triangulated subcategory of  $\mathcal{D}$ . The following assertions are equivalent:*

- 1)  $(\mathcal{D}' \cap \mathcal{U}, \mathcal{D}' \cap \mathcal{V}[1])$  is a t-structure on  $\mathcal{D}'$ .
- 2)  $(u\tau_{\mathcal{U}})(\mathcal{D}') \subseteq \mathcal{D}'$ .

**4.2. ‘Right bounded’ triangulated subcategories.** We present now a very particular situation in which condition 2) of the lemma above can be improved. As in subsection 3.2, let  $\mathcal{D}$  be a triangulated category with small coproducts, and let  $\mathcal{Q}$  be a set of objects of  $\mathcal{D}$  such that  $\text{Susp}(\mathcal{Q})$  is an aisle in  $\mathcal{D}$  (and so  $\text{Susp}(\mathcal{Q}) = \text{aisle}(\mathcal{Q})$ ). Let  $\mathcal{D}' := \bigcup_{n \in \mathbb{Z}} \text{aisle}(\mathcal{Q})[n]$ .

**Proposition 3.** *Assume that  $\mathcal{Q}$  is a set of perfect generators of  $\mathcal{D}$  such that  $\text{aisle}(\mathcal{Q})^\perp$  is closed under small coproducts. Let  $(\mathcal{U}, \mathcal{V})$  be a t-structure on  $\mathcal{D}$  such that  $\mathcal{U}$  is triangulated and  $\mathcal{V}$  is closed under coproducts. The following assertions are equivalent:*

- 1)  $(\mathcal{D}' \cap \mathcal{U}, \mathcal{D}' \cap \mathcal{V})$  is a t-structure on  $\mathcal{D}'$ .
- 2)  $(u\tau_{\mathcal{U}})(\mathcal{Q}) \subseteq \text{aisle}(\mathcal{Q})[n]$  for some integer  $n$ .

*Proof.* 1)  $\Rightarrow$  2) Thanks to Lemma 3, it suffices to prove that  $(u\tau_{\mathcal{U}})(\mathcal{D}') \subseteq \mathcal{D}'$  implies condition 2) of the proposition. Since  $N := \coprod_{Q \in \mathcal{Q}} Q$  belongs to  $\text{aisle}(\mathcal{Q})$ , there exists an integer  $n$  such that  $u\tau_{\mathcal{U}}N$  belongs to  $\text{aisle}(\mathcal{Q})[n]$ . Now notice that for each  $Q \in \mathcal{Q}$  we have that  $u\tau_{\mathcal{U}}Q$  is a direct summand of  $u\tau_{\mathcal{U}}N$ . But since  $\text{aisle}(\mathcal{Q})[n]$  is closed under Milnor colimits in  $\mathcal{D}$  then it is also closed under direct summands. This implies that  $u\tau_{\mathcal{U}}Q$  belongs to  $\text{aisle}(\mathcal{Q})[n]$ . 2)  $\Rightarrow$  1) Thanks to Lemma 3, it suffices to prove the inclusion  $(u\tau_{\mathcal{U}})(\mathcal{D}') \subseteq \mathcal{D}'$ . Let  $N$  be an object of  $\mathcal{D}'$  and fix an integer  $i$  such that  $N[i]$  belongs to  $\text{aisle}(\mathcal{Q})$ . The proof of [27, Theorem 2.2] shows us that  $N[i]$  is the Milnor colimit of a sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$$

where  $M_n \in \text{Sum}(\mathcal{Q}^+)^{*n}$  for each  $n \geq 0$ . Now, since  $u\tau_{\mathcal{U}}$  commutes with small coproducts, by applying it to the corresponding Milnor triangle we get that  $u\tau_{\mathcal{U}}N[i]$  belongs to  $\mathcal{D}'$ , and then so does  $u\tau_{\mathcal{U}}N$ .

✓

**Remark 2.** If in Proposition 3 the set  $\mathcal{Q}$  is finite, then one can replace condition 2) by:  $(u\tau_{\mathcal{U}})(\mathcal{Q}) \subseteq \mathcal{D}'$ .

**Corollary 2.** *Assume that  $\mathcal{Q}$  is a set of perfect generators of  $\mathcal{D}$  such that  $\text{aisle}(\mathcal{Q})^\perp$  is closed under small coproducts. The following assertions are equivalent for a TTF triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  on  $\mathcal{D}$ :*

- 1)  $(\mathcal{D}' \cap \mathcal{X}, \mathcal{D}' \cap \mathcal{Y}, \mathcal{D}' \cap \mathcal{Z})$  is a TTF triple on  $\mathcal{D}'$ .
- 2) The following conditions hold:
  - 2.1)  $(x\tau_{\mathcal{X}})(\mathcal{Q}) \subseteq \text{aisle}(\mathcal{Q})[n]$  for some integer  $n$ .
  - 2.2)  $(y\tau_{\mathcal{Y}})(\mathcal{D}') \subseteq \mathcal{D}'$ .

**Example 3.** Let  $I$  be a two-sided ideal of a  $k$ -algebra  $A$ , and assume the canonical projection  $\pi : A \rightarrow A/I$  is a *homological epimorphism* in the sense of W. Geigle and H. Lenzing [7]. We know (cf. [24, Example 5.3.4]) that in this case  $\mathcal{D}A$  is a recollement of  $\mathcal{D}(A/I)$  and  $\text{Tri}_{\mathcal{D}A}(I)$ :

$$\begin{array}{ccccc}
 & \overset{? \otimes_A^{\mathbf{L}} A/I}{\curvearrowright} & & \overset{? \otimes_A^{\mathbf{L}} I}{\curvearrowright} & \\
 \mathcal{D}(A/I) & \xrightarrow{\pi_*} & \mathcal{D}A & \xrightarrow{\quad} & \text{Tri}_{\mathcal{D}A}(I) \\
 & \underset{\mathbf{R} \text{Hom}_A(A/I, ?)}{\curvearrowleft} & & \underset{x}{\curvearrowleft} & 
 \end{array}$$

where  $x$  is the inclusion functor. Let  $\mathcal{C}_{dg}A$  be the dg category whose objects are the complexes of  $A$ -modules and whose morphisms are given by complexes of  $k$ -modules  $\mathcal{C}_{dg}(A)(L, M)$ , with  $n$ th component formed by the morphisms of  $\mathbf{Z}$ -graded  $k$ -modules homogeneous of degree  $n$  and with differential given by the commutator  $d(f) = d_M f - (-1)^{|f|} f d_L$ , where  $|f|$  is the degree of  $f$ . Notice that the corresponding category of 0-cocycles  $\mathbf{Z}^0(\mathcal{C}_{dg}A)$  is the category  $\mathcal{C}A$  of complexes of  $A$ -modules and the corresponding category of 0-cohomology  $H^0(\mathcal{C}_{dg}A)$  is the category  $\mathcal{H}A$  of complexes of  $A$ -modules up to homotopy. In case  $I$  is compact in  $\text{Tri}_{\mathcal{D}A}(I)$ , the proof of [11, Theorem 4.3] implies that  $\mathcal{D}A$  is a recollement of  $\mathcal{D}(A/I)$  and  $\mathcal{D}C$ , where  $C$  is the dg algebra  $(\mathcal{C}_{dg}A)(\mathbf{i}I, \mathbf{i}I)$  and  $\mathbf{i} : \mathcal{D}A \rightarrow \mathcal{H}A$  is the *fibrant replacement functor* (cf. [13]). Indeed, the dg  $A$ - $C$ -bimodule  $\mathbf{i}I$  induces mutually quasi-inverse triangle equivalences

$$\begin{array}{ccc}
 & \mathbf{R} \text{Hom}_A(\mathbf{i}I, ?) & \\
 \text{Tri}_{\mathcal{D}A}(I) & \xrightleftharpoons[? \otimes_C^{\mathbf{L}} \mathbf{i}I]{} & \mathcal{D}C.
 \end{array}$$

Thanks to Corollary 2, we know that the associated TTF triple restricts to  $\mathcal{D}^-A$  if and only if the following conditions hold:

- 1)  $A \otimes_A^{\mathbf{L}} I \cong I$  belongs to  $\mathcal{D}^-A$ ,
- 2)  $\mathbf{R} \text{Hom}_A(A/I, M)$  belongs to  $\mathcal{D}^-(A/I)$  for each  $M$  in  $\mathcal{D}^-A$ .

Of course, the first condition always holds. Thanks to S. König's criterion explained at the beginning of the proof of [16, Theorem 1], we have that the second condition holds if and only if  $A/I$  has finite projective dimension regarded as a right  $A$ -module or, equivalently,  $I$  has finite projective dimension regarded as a right  $A$ -module. Assume then that  $I_A$  has finite projective dimension and also that it is compact in  $\text{Tri}_{\mathcal{D}A}(I)$ . In this case the mutually quasi-inverse triangle equivalences between  $\text{Tri}_{\mathcal{D}A}(I)$  and  $\mathcal{D}C$  restrict to mutually quasi-inverse triangle equivalences

$$\text{Tri}_{\mathcal{D}A}(I) \cap \mathcal{D}^-A \xrightleftharpoons{\quad} \mathcal{D}^-C.$$

Therefore,  $\mathcal{D}^-A$  is a recollement of  $\mathcal{D}^-(A/I)$  and  $\mathcal{D}^-C$ . This example contains as particular cases the recollement data of Corollary 11, Corollary 12 and Corollary 15 of [16], and describes functors appearing in those recollement data as restrictions of total derived functors.

## 5. RECOLLEMENT OF RIGHT BOUNDED DERIVED CATEGORIES

All through this section the appearing dg categories are small.

## 5.1. Bounds.

**Definition 3.** Let  $\mathcal{A}$  be a dg category. Consider the corresponding dg category  $\mathcal{C}_{dg}\mathcal{A}$  (cf. [13]), which is the ‘dg generalization to several objects’ of the dg category  $\mathcal{C}_{dg}A$  associated to an algebra  $A$  appearing in Example 3. A *fibrant replacement* of a set  $\mathcal{P}$  of objects of the derived category  $\mathcal{DA}$  is a full subcategory  $\mathcal{B}$  of  $\mathcal{C}_{dg}\mathcal{A}$  formed by the fibrant replacements  $\mathbf{i}P$ , in the sense of [13], of the modules  $P$  of  $\mathcal{P}$ .

Notice that  $\mathcal{B}$  is a dg category and we have a dg  $\mathcal{B}$ - $\mathcal{A}$ -bimodule  $X$  defined by  $X(A, B) := B(A)$  for  $A$  in  $\mathcal{A}$  and  $B$  in  $\mathcal{B}$ . It is well-known (cf. [11, 13]) that this gives rise to a functor

$$\mathcal{H}om_{\mathcal{A}}(X, ?) : \mathcal{C}_{dg}\mathcal{A} \rightarrow \mathcal{C}_{dg}\mathcal{B}$$

which induces triangle functors

$$\mathcal{H}om_{\mathcal{A}}(X, ?) : \mathcal{HA} \rightarrow \mathcal{HB}$$

and

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(X, ?) : \mathcal{DA} \rightarrow \mathcal{DB}.$$

**Definition 4.** Under the conditions above, we say that:

- 1)  $\mathcal{P}$  is *right bounded* if  $\mathcal{P} \subseteq \mathcal{D}^{\leq n}\mathcal{A}$  for some  $n \in \mathbf{Z}$ .
- 2)  $\mathcal{P}$  is *dually right bounded* if the functor

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(X, ?) : \mathcal{DA} \rightarrow \mathcal{DB}$$

sends an object of  $\mathcal{D}^-\mathcal{A}$  to an object of  $\mathcal{D}^-\mathcal{B}$ .

*A priori*, the notion of “dually right bounded” depends on the fibrant replacement of  $\mathcal{P}$ , however this is not really a problem for our purposes. In the subsequent propositions we will present the two situations in which we are most interested, where the notion of “dually right bounded” is independent of the fibrant replacement.

**Proposition 4.** Let  $\mathcal{A}$  be a dg category and  $\mathcal{P}$  a set of objects of  $\mathcal{D}^-\mathcal{A}$ . Assume that there exists an integer  $m$  such that for every two objects  $P$  and  $P'$  of  $\mathcal{P}$  we have  $(\mathcal{DA})(P, P'[i]) = 0$  for  $i > m$ . Consider the following assertions:

- 1)  $\mathcal{P}$  is *dually right bounded*.
- 2) For each object  $M$  of  $\mathcal{D}^-\mathcal{A}$  there exists an integer  $s_M$  such that  $(\mathcal{DA})(P, M[i]) = 0$  for every  $P \in \mathcal{P}$  and every  $i > m + s_M$ .

Then 1) implies 2) and, if  $m = 0$ , we also have that 2) implies 1).

*Proof.* Let  $\mathcal{B}$  be a fibrant replacement of the set  $\mathcal{P}$ . Notice that the assumption on the set  $\mathcal{P}$  is equivalent to say that  $\mathcal{B}$  has cohomology concentrated in degrees  $(-\infty, m]$ . Let  $X$  be the associated  $\mathcal{A}$ - $\mathcal{B}$ -bimodule. Assertion 1) says that for each  $M \in \mathcal{D}^-\mathcal{A}$  there exists an integer  $s_M$  such that

$$(\mathcal{C}_{dg}\mathcal{A})(?, \mathbf{i}M)|_{\mathcal{B}} \in \mathcal{D}^{\leq s_M}\mathcal{B}.$$

Now, thanks to Lemma 1, this implies that  $(\mathcal{DA})(P, M[i]) = 0$  for every  $P \in \mathcal{P}$  and every  $i > m + s_M$ . As stated in Lemma 1, in case  $m = 0$  we can go backward in the proof. √

By using S. König's criterion which characterizes the bounded complexes of projective modules inside the right bounded derived category of an algebra (see the beginning of the proof of [16, Theorem 1]), we deduce the following:

**Corollary 3.** *Let  $A$  be an ordinary algebra, and let  $P$  an object of the right bounded derived category  $\mathcal{D}^-A$  of  $A$  such that  $(\mathcal{D}A)(P, P[i]) = 0$  for  $i \geq 1$ . Then  $P$  is dually right bounded if and only if it is quasi-isomorphic to a bounded complex of projective  $A$ -modules.*

**Proposition 5.** *Let  $\mathcal{A}$  be a dg category and let  $\mathcal{P}$  be a set of objects of  $\mathcal{D}^- \mathcal{A}$  such that:*

- a) *it is right bounded,*
- b) *its objects are compact in  $\text{Tri}_{\mathcal{D}\mathcal{A}}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$ ,*
- c)  *$\text{Tri}_{\mathcal{D}\mathcal{A}}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is exhaustively generated to the left by  $\mathcal{P}$ .*

*Let  $\mathcal{B}$  be a fibrant replacement of  $\mathcal{P}$  and let  $X$  be the associated  $\mathcal{B}$ - $\mathcal{A}$ -bimodule. Then, the functor  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}\mathcal{B} \rightarrow \mathcal{D}\mathcal{A}$  induces a triangle equivalence*

$$? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}^- \mathcal{B} \xrightarrow{\sim} \text{Tri}_{\mathcal{D}\mathcal{A}}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A},$$

*and the following assertions are equivalent:*

- 1)  *$\text{Tri}_{\mathcal{D}\mathcal{A}}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is an aisle in  $\mathcal{D}^- \mathcal{A}$ .*
- 2)  *$\mathcal{P}$  is dually right bounded.*

*Proof. First step: The triangle functor*

$$? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}\mathcal{B} \rightarrow \mathcal{D}\mathcal{A}$$

*induces a triangle functor*

$$? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}^- \mathcal{B} \rightarrow \text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}.$$

Let  $\mathcal{U}$  be the full subcategory of  $\mathcal{D}^- \mathcal{B}$  formed by those  $N$  such that  $N \otimes_{\mathcal{B}}^{\mathbf{L}} X \in \text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$ . It is a full triangulated subcategory of  $\mathcal{D}^- \mathcal{B}$ . Notice that, if  $B = \mathbf{i}P$  is the object of  $\mathcal{B}$  corresponding to a certain  $P \in \mathcal{P}$ , then

$$B^\wedge \otimes_{\mathcal{B}}^{\mathbf{L}} X \cong \mathbf{i}P \cong P \in \text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}.$$

This proves that  $\mathcal{U}$  contains the representable dg  $\mathcal{B}$ -modules  $B^\wedge$ . It also proves that, since  $\text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is closed under small coproducts of finite extensions of objects  $\text{Sum}(\mathcal{P}^+)$ , then  $\mathcal{U}$  is closed under small coproducts of finite extensions of objects of  $\text{Sum}(\{B^\wedge\}_{B \in \mathcal{B}}^+)$ . Since  $\mathcal{D}^- \mathcal{B}$  is exhaustively generated to the left by the representable modules  $B^\wedge$ ,  $B \in \mathcal{B}$ , this implies that  $\mathcal{U} = \mathcal{D}^- \mathcal{B}$ . *Second step: The functor  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}^- \mathcal{B} \rightarrow \text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is a triangle equivalence.* To prove it we will use the techniques of [11, Lemma 4.2]. If  $B = \mathbf{i}P$  is the object of  $\mathcal{B}$  corresponding to  $P \in \mathcal{P}$ , we have seen already that  $B^\wedge \otimes_{\mathcal{B}}^{\mathbf{L}} X \cong P$ , which is compact in  $\text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  by hypothesis. Also, if  $B = \mathbf{i}P$  and  $B' = \mathbf{i}P'$  are objects of  $\mathcal{B}$ , we have

$$\begin{aligned} (\mathcal{D}\mathcal{B})(B^\wedge, B'^\wedge[n]) &\xrightarrow{\sim} H^n \mathcal{B}(B, B') = \\ &= (\mathcal{H}\mathcal{A})(\mathbf{i}P, \mathbf{i}P'[n]) \xrightarrow{\sim} (\mathcal{D}\mathcal{A})(B^\wedge \otimes_{\mathcal{B}}^{\mathbf{L}} X, B'^\wedge[n] \otimes_{\mathcal{B}}^{\mathbf{L}} X). \end{aligned}$$

Let  $\mathcal{U}$  be the full subcategory of  $\mathcal{D}^- \mathcal{B}$  formed by those objects  $N$  such that  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X$  induces an isomorphism

$$(\mathcal{D}^- \mathcal{B})(B^\wedge[n], N) \xrightarrow{\sim} (\mathcal{D}^- \mathcal{A})(B^\wedge[n] \otimes_{\mathcal{B}}^{\mathbf{L}} X, N \otimes_{\mathcal{B}}^{\mathbf{L}} X)$$

for each  $B \in \mathcal{B}$  and each  $n \in \mathbf{Z}$ . It is a full triangulated subcategory of  $\mathcal{D}^-\mathcal{B}$  closed under small coproducts and containing the representable modules  $B^\wedge$ ,  $B \in \mathcal{B}$ . Since  $\mathcal{D}^-\mathcal{B}$  is exhaustively generated to the left by the representable modules  $B^\wedge$ ,  $B \in \mathcal{B}$ , this implies that  $\mathcal{U} = \mathcal{D}^-\mathcal{B}$ . Fix now an object  $N \in \mathcal{D}^-\mathcal{B}$  and consider the full subcategory  $\mathcal{V}$  of  $\mathcal{D}^-\mathcal{B}$  formed by the objects  $M$  such that  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X$  induces an isomorphism

$$(\mathcal{D}^-\mathcal{B})(M, N[n]) \xrightarrow{\sim} (\mathcal{D}^-\mathcal{B})(M \otimes_{\mathcal{B}}^{\mathbf{L}} X, N[n] \otimes_{\mathcal{B}}^{\mathbf{L}} X)$$

for each  $N \in \mathbf{Z}$ . Again, it is a full triangulated subcategory of  $\mathcal{D}^-\mathcal{B}$  containing the representable modules and closed under small coproducts, which implies that  $\mathcal{V} = \mathcal{DB}$ . Therefore, we have already proved that  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X$  is fully faithful. Finally, by hypothesis,  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^-\mathcal{A}$  is exhaustively generated to the left by the objects of  $\mathcal{P}$ . Since they are in the essential image of the functor  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X$ , we deduce that it is essentially surjective. *Third step: Thanks to the second step, 1) holds if and only if the functor  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X : \mathcal{D}^-\mathcal{B} \rightarrow \mathcal{D}^-\mathcal{A}$  has a right adjoint. Let us prove that this happens if and only if  $\mathcal{P}$  is dually right bounded.* The ‘if’ part is clear. Conversely, let  $G : \mathcal{D}^-\mathcal{A} \rightarrow \mathcal{D}^-\mathcal{B}$  be a right adjoint to  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X$ . For simplicity, put  $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(X, ?) = H_X$ . Consider the diagram

$$\begin{array}{ccccc} & & \mathcal{D}^-\mathcal{B} & \xrightleftharpoons[? \otimes_{\mathcal{B}}^{\mathbf{L}} X]{? \otimes_{\mathcal{B}}^{\mathbf{L}} X} & \mathcal{D}^-\mathcal{A} \\ & \nearrow \iota & \downarrow \iota_{\mathcal{B}} & \xleftarrow{G} & \downarrow \iota_{\mathcal{A}} \\ \mathcal{D}^{\leq n}\mathcal{B} & \xrightleftharpoons[\tau^{\leq n}]{\iota} & \mathcal{DB} & \xrightleftharpoons[H_X]{? \otimes_{\mathcal{B}}^{\mathbf{L}} X} & \mathcal{DA} \end{array}$$

where  $n$  is any integer and  $\iota$  is the inclusion functor. We have that

$$\tau^{\leq n} \circ \iota_{\mathcal{B}} \circ G \cong \tau^{\leq n} \circ H_X \circ \iota_{\mathcal{A}}$$

since these two compositions are right adjoint to  $? \otimes_{\mathcal{B}}^{\mathbf{L}} X \circ \iota$ . Let the  $M$  be an object of  $\mathcal{D}^-\mathcal{A}$  and fix an integer  $n$  such that  $GM \in \mathcal{D}^{\leq n}\mathcal{B}$ . Then, we get

$$\tau^{\leq n} H_X(M) \cong \tau^{\leq n} G(M) \cong \tau^{\leq n+i} G(M) \cong \tau^{\leq n+i} H_X(M)$$

for each  $i \geq 0$ . This implies that  $\tau^{>n}(H_X M) \in \mathcal{D}^{>n+i}\mathcal{B}$  for each  $i \geq 0$ . In particular,

$$H^j(\tau^{>n} H_X(M)) = 0$$

for every  $j \in \mathbf{Z}$ , that is to say,  $\tau^{>n}(H_X(M)) = 0$ . Thus,  $H_X(M) \in \mathcal{D}^{\leq n}\mathcal{B}$ .  $\checkmark$

## 5.2. Recollement of general right bounded derived categories.

**Theorem 1.** *Let  $\mathcal{A}$  be a dg category. The following assertions are equivalent:*

- 1)  $\mathcal{D}^-\mathcal{A}$  is a recollement of  $\mathcal{D}^-\mathcal{B}$  and  $\mathcal{D}^-\mathcal{C}$ , for certain dg categories  $\mathcal{B}$  and  $\mathcal{C}$ .
- 2) There exist sets  $\mathcal{P}$ ,  $\mathcal{Q}$  in  $\mathcal{D}^-\mathcal{A}$  such that:
  - 2.1)  $\mathcal{P}$  and  $\mathcal{Q}$  are right bounded.
  - 2.2)  $\mathcal{P}$  and  $\mathcal{Q}$  are dually right bounded.
  - 2.3)  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^-\mathcal{A}$  is exhaustively generated to the left by  $\mathcal{P}$  and the objects of  $\mathcal{P}$  are compact in  $\mathcal{DA}$ .
  - 2.4)  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^-\mathcal{A}$  is exhaustively generated to the left by  $\mathcal{Q}$  and the objects of  $\mathcal{Q}$  are compact in  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^-\mathcal{A}$ .
  - 2.5)  $(\mathcal{DA})(P[i], Q) = 0$  for each  $P \in \mathcal{P}$ ,  $Q \in \mathcal{Q}$  and  $i \in \mathbf{Z}$ .
  - 2.6)  $\mathcal{P} \cup \mathcal{Q}$  generates  $\mathcal{DA}$ .



*Proof.* 1)  $\Rightarrow$  2) Consider the décollement

$$\begin{array}{ccccc} & i_* & & j_* & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{D}^- \mathcal{B} & \xrightarrow{i_* = i_!} & \mathcal{D}^- \mathcal{A} & \xrightarrow{j_* = j_!} & \mathcal{D}^- \mathcal{C}, \\ & \curvearrowleft & & \curvearrowleft & \\ & i^! & & j^! & \end{array}$$

and let  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  be the corresponding TTF triple on  $\mathcal{D}^- \mathcal{A}$ . Let  $\mathcal{P}$  be the set formed by all the objects  $j_!(C^\wedge)$ ,  $C \in \mathcal{C}$ , and let  $\mathcal{Q}$  be the set formed by all the objects  $i_*(B^\wedge)$ ,  $B \in \mathcal{B}$ . 2.1) Notice that the coproduct  $\coprod_{C \in \mathcal{C}} C^\wedge$  lives in  $\mathcal{D}^- \mathcal{C}$  and, since  $j_! : \mathcal{D}^- \mathcal{C} \xrightarrow{\sim} \mathcal{X}'$  is a triangle equivalence, then there exists in  $\mathcal{D}^- \mathcal{A}$  the coproduct  $\coprod_{P \in \mathcal{P}} P$ . Now, the claim in the proof of Lemma 2 implies that  $\mathcal{P}$  is right bounded. Similarly for  $\mathcal{Q}$ . 2.3) Since  $j_! : \mathcal{D}^- \mathcal{C} \xrightarrow{\sim} \mathcal{X}'$  is a triangle equivalence, then  $\mathcal{X}'$  is exhaustively generated to the left by the set  $\mathcal{P}$ , whose objects are compact in  $\mathcal{X}'$ . Then Proposition 2 says that  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  is the restriction of a TTF triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  on  $\mathcal{DA}$ . Moreover,  $\mathcal{X} = \text{Tria}(\mathcal{P})$  and so  $\mathcal{X}' = \text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$ . This proves that  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is exhaustively generated to the left by  $\mathcal{P}$ . By using that  $\mathcal{X}'$  is an aisle in  $\mathcal{D}^- \mathcal{A}$  and that  $\mathcal{Y}'$  is closed under small coproducts in  $\mathcal{D}^- \mathcal{A}$ , we can prove that the objects of  $\mathcal{P}$  are compact in  $\mathcal{D}^- \mathcal{A}$ . Finally, Lemma 2 implies that they are also compact in  $\mathcal{DA}$ . 2.4) Since  $i_* : \mathcal{D}^- \mathcal{B} \xrightarrow{\sim} \mathcal{Y}'$  is a triangle equivalence, then  $\mathcal{Y}'$  is exhaustively generated to the left by the set  $\mathcal{Q}$ , whose objects are compact in  $\mathcal{Y}'$ . From the proof of 2.3) we know that

$$\mathcal{Y}' = \text{Tria}(\mathcal{P})^\perp \cap \mathcal{D}^- \mathcal{A}.$$

Of course,  $\mathcal{Q}$  is contained in  $\mathcal{Y}'$  and so  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A}$  is contained in  $\mathcal{Y}'$ . Notice that  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A}$  is a full triangulated subcategory of  $\mathcal{Y}'$  containing  $\mathcal{Q}$  and closed under small coproducts of objects of  $\bigcup_{n \geq 0} \text{Sum}(\mathcal{Q}^+)^{*n}$ . Since  $\mathcal{Y}'$  is exhaustively generated to the left by the set  $\mathcal{Q}$ , this implies that  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A} = \mathcal{Y}'$ . 2.2) From the proof of 2.3), we know that  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is an aisle in  $\mathcal{D}^- \mathcal{A}$ . Then Proposition 5 implies that  $\mathcal{P}$  is dually right bounded. Similarly for  $\mathcal{Q}$ . 2.5) and 2.6) follow from the fact that  $(\text{Tria}(\mathcal{P}), \text{Tria}(\mathcal{Q}))$  is a t-structure on  $\mathcal{DA}$ . 2)  $\Rightarrow$  1) Since the objects of  $\mathcal{P}$  are compact in  $\mathcal{DA}$ , then Brown representability theorem implies that  $(\mathcal{X}, \mathcal{Y}) := (\text{Tria}(\mathcal{P}), \text{Tria}(\mathcal{P})^\perp)$  is a t-structure on  $\mathcal{DA}$ . Notice that  $\mathcal{Y}$  is closed under small coproducts and that  $\tau_{\mathcal{Y}}$  takes a set of compact generators of  $\mathcal{DA}$  to a set of compact generators of  $\mathcal{Y}$ . Then,  $\mathcal{Y}$  is a compactly generated triangulated category and Brown representability theorem implies that it is an aisle. Therefore,

$$(\text{Tria}(\mathcal{P}), \text{Tria}(\mathcal{P})^\perp, (\text{Tria}(\mathcal{P})^\perp)^\perp)$$

is a TTF triple on  $\mathcal{DA}$ . From conditions 2.5) and 2.6) we deduce that  $\mathcal{Q}$  generates  $\text{Tria}(\mathcal{P})^\perp$ . Moreover, since  $\text{Tria}(\mathcal{P})^\perp$  is closed under small coproducts, then  $\text{Tria}(\mathcal{Q})$  is contained in  $\text{Tria}(\mathcal{P})^\perp$ . It is an exercise to prove that the fact that  $\text{Tria}(\mathcal{Q})$  is an aisle in  $\mathcal{DA}$  (cf. [24, Corollary 4.6.10], [22, Corollary 3.2], [25, Corollary 3.12]) implies that  $\text{Tria}(\mathcal{P})^\perp = \text{Tria}(\mathcal{Q})$ . Proposition 5 tells us that  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  and  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A}$  are aisles in  $\mathcal{D}^- \mathcal{A}$ . Given  $M \in \mathcal{D}^- \mathcal{A}$ , consider the triangle

$$M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$$

in  $\mathcal{D}^- \mathcal{A}$  with  $M' \in \text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  and  $M'' \in (\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A})^\perp$ . In particular,  $M' \in \text{Tria}(\mathcal{P})$  and  $M'' \in \text{Tria}(\mathcal{P})^\perp = \text{Tria}(\mathcal{Q})$ . This proves that

$$(\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}, \text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A})$$



is a t-structure on  $\mathcal{D}^- \mathcal{A}$ . Similarly,

$$(\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A}, \text{Tria}(\mathcal{Q})^\perp \cap \mathcal{D}^- \mathcal{A})$$

is a t-structure on  $\mathcal{D}^- \mathcal{A}$ . These t-structures together form a TTF triple  $(\mathcal{X}', \mathcal{Y}', \mathcal{Z}')$  on  $\mathcal{D}^- \mathcal{A}$ . Finally, Proposition 5 implies that  $\mathcal{X}' \cong \mathcal{D}^- \mathcal{C}$  (for a fibrant replacement  $\mathcal{C}$  of  $\mathcal{P}$ ) and  $\mathcal{Y}' \cong \mathcal{D}^- \mathcal{B}$  (for a fibrant replacement  $\mathcal{B}$  of  $\mathcal{Q}$ ).  $\checkmark$

We will prove in Corollary 5 below that conditions of assertion 2 in Theorem 1 can be weakened under certain extra hypotheses. But first we need some preliminary results. The following one is a ‘right bounded’ version of the proof of B. Keller’s theorem [11, Theorem 5.2]:

**Proposition 6.** *Let  $\mathcal{P}$  be a set of objects of a triangulated category  $\mathcal{D}$  such that*

- 1) *the objects of  $\mathcal{P}$  are compact in  $\text{Tria}(\mathcal{P})$ ,*
- 2)  *$\mathcal{D}(P, P'[i]) = 0$  for each  $P, P' \in \mathcal{P}$  and  $i \geq 1$ ,*
- 3) *small coproducts of finite extensions of objects of  $\text{Sum}(\mathcal{P}^+)$  exist in  $\mathcal{D}$ ,*
- 4) *for each  $M \in \mathcal{D}$  there exists  $k_M \in \mathbf{Z}$  such that  $\mathcal{D}(P[n], M) = 0$  for all  $n < k_M$  and  $P \in \mathcal{P}$ .*

*Then  $\text{Tria}(\mathcal{P})$  is an aisle in  $\mathcal{D}$  exhaustively generated to the left by  $\mathcal{P}$ . In particular, if  $\mathcal{P}$  generates  $\mathcal{D}$ , then  $\text{Tria}(\mathcal{P}) = \mathcal{D}$ .*

*Proof.* We include the proof for the sake of completeness. Let  $M \in \mathcal{D}$ . We know that if  $\mathcal{D}(P[n], M) \neq 0$  for some  $P \in \mathcal{P}$ , then  $n \geq k_M$ . Since  $\mathcal{P}$  is a set, there exists an object

$$P_0 \in \text{Sum}(\mathcal{P}^+[k_M])$$

and a morphism  $\pi_0 : P_0 \rightarrow M$  inducing a surjection

$$\pi_0^\wedge : \mathcal{D}(P[n], P_0) \rightarrow \mathcal{D}(P[n], M)$$

for each  $P \in \mathcal{P}$ ,  $n \in \mathbf{Z}$ . Indeed, one can take

$$P_0 := \coprod_{P \in \mathcal{P}, n \geq k_M} P[n]^{(\mathcal{D}(P[n], M))}.$$

Now, we will inductively construct a commutative diagram

$$\begin{array}{ccccccc} P_0 & \xrightarrow{f_0} & P_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_q} & P_q \xrightarrow{f_q} \cdots, q \geq 0 \\ & \searrow \pi_0 & \downarrow \pi_1 & & & \nearrow \pi_q & \\ & & M & & & & \end{array}$$

such that:

- a)  $P_q \in \text{Sum}(\mathcal{P}^+[k_M])^{*q}$ ,
- b)  $\pi_q$  induces a surjection

$$\pi_q^\wedge : \mathcal{D}(P[n], P_q) \rightarrow \mathcal{D}(P[n], M)$$

for each  $P \in \mathcal{P}$ ,  $n \in \mathbf{Z}$ .

Suppose for some  $q \geq 0$  we have constructed  $P_q$  and  $\pi_q$ . Consider the triangle

$$C_q \xrightarrow{\alpha_q} P_q \xrightarrow{\pi_q} M \rightarrow C_q[1]$$

induced by  $\pi_q$ . By applying  $\mathcal{D}(P[n], ?)$  we get a long exact sequence

$$\cdots \rightarrow \mathcal{D}(P[n+1], M) \rightarrow \mathcal{D}(P[n], C_q) \rightarrow \mathcal{D}(P[n], P_q) \rightarrow \cdots$$

If  $\mathcal{D}(P[n], C_q) \neq 0$ , then either  $\mathcal{D}(P[n+1], M) \neq 0$  or  $\mathcal{D}(P[n], P_q) \neq 0$ . In the first case, we would have  $n \geq k_M - 1$ . In the second case we would have  $\mathcal{D}(P[n], P'[m]) \neq 0$  for some  $P' \in \mathcal{P}$ ,  $m \geq k_M$ , and so  $n \geq m \geq k_M$ . Therefore,  $\mathcal{D}(P[n], C_q) \neq 0$  implies  $n \geq k_M - 1$ . This allows us to take

$$Z_q \in \text{Sum}(\mathcal{P}^+[k_M - 1])$$

together with a morphism  $\beta_q : Z_q \rightarrow C_q$  inducing a surjection

$$\beta_q^\wedge : \mathcal{D}(P[n], Z_q) \rightarrow \mathcal{D}(P[n], C_q)$$

for each  $P \in \mathcal{P}$ ,  $n \in \mathbf{Z}$ . Define  $f_q$  by the triangle

$$Z_q \xrightarrow{\alpha_q \beta_q} P_q \xrightarrow{f_q} P_{q+1} \rightarrow Z_q[1]$$

Since  $\pi_q \alpha_q = 0$ , there exists  $\pi_{q+1} : P_{q+1} \rightarrow M$  such that  $\pi_{q+1} f_q = \pi_q$ . Notice that, since

$$Z_q[1] \in \text{Sum}(\mathcal{P}^+[k_M]),$$

then

$$P_{q+1} \in \text{Sum}(\mathcal{P}^+[k_M])^{*(q+1)}.$$

Also, the surjectivity required for  $\pi_{q+1}^\wedge$  follows from the surjectivity guaranteed for  $\pi_q^\wedge$ . Define  $P_\infty$  to be the Milnor colimit of the sequence  $f_q$ ,  $q \geq 0$ :

$$\coprod_{q \geq 0} P_q \xrightarrow{\varphi} \coprod_{q \geq 0} P_q \xrightarrow{\psi} P_\infty \rightarrow \coprod_{q \geq 0} P_q[1].$$

Consider the morphism

$$\theta = \begin{bmatrix} \pi_0 & \pi_1 & \dots \end{bmatrix} : \coprod_{q \geq 0} P_q \rightarrow M.$$

Since  $\pi_{q+1} f_q = \pi_q$  for every  $q \geq 0$ , we have  $\theta \varphi = 0$ , which induces a morphism  $\pi_\infty : P_\infty \rightarrow M$  such that  $\pi_\infty \psi = \theta$ . If we prove that  $\pi_\infty$  induces an isomorphism

$$\pi_\infty^\wedge : \mathcal{D}(P[n], P_\infty) \xrightarrow{\sim} \mathcal{D}(P[n], M)$$

for every  $P \in \mathcal{P}$ ,  $n \in \mathbf{Z}$ , then we would have

$$\mathcal{D}(P[n], \text{Cone}(\pi_\infty)) = 0$$

for every  $P \in \mathcal{P}$ ,  $n \in \mathbf{Z}$ , that is to say

$$\text{Cone}(\pi_\infty) \in \text{Tria}(\mathcal{P})^\perp.$$

Therefore, we would have proved that  $\text{Tria}(\mathcal{P})$  is an aisle in  $\mathcal{D}$ . Also, if  $M \in \text{Tria}(\mathcal{P})$ , in the triangle

$$P_\infty \xrightarrow{\pi_\infty} M \rightarrow \text{Cone}(\pi_\infty) \rightarrow P_\infty[1]$$

we would have that  $P_\infty, M \in \text{Tria}(\mathcal{P})$ , which implies

$$\text{Cone}(\pi_\infty) \in \text{Tria}(\mathcal{P}).$$

Therefore,  $\text{Cone}(\pi_\infty) = 0$  and so  $\pi_\infty$  is an isomorphism. Thus, we would have proved that for every object of  $\text{Tria}(\mathcal{P})$  there exists an integer  $k_M$  and a triangle

$$\coprod_{q \geq 0} P_q \rightarrow \coprod_{q \geq 0} P_q \rightarrow M[-k_M] \rightarrow \coprod_{q \geq 0} P_q[1]$$

with  $P_q \in \text{Sum}(\mathcal{P}^+)^{*q}$ ,  $q \geq 0$ . In particular, we would have that  $\text{Tria}(\mathcal{P})$  is exhaustively generated to the left by  $\mathcal{P}$ . Let us prove the bijectivity of  $\pi_\infty^\wedge$ . The surjectivity follows from the identity  $\pi_\infty^\wedge \psi^\wedge = \theta^\wedge$  and the fact that  $\theta^\wedge$  is surjective

(thanks to the surjectivity of the  $\pi_q^\wedge$ ,  $q \geq 0$  and the compactness of the  $P \in \mathcal{P}$ ). Now consider the commutative diagram

$$\begin{array}{ccccc} \coprod_{q \geq 0} \mathcal{D}(P[n], P_q) & \xrightarrow{\varphi^\wedge} & \coprod_{q \geq 0} \mathcal{D}(P[n], P_q) & \xrightarrow{\psi^\wedge} & \mathcal{D}(P[n], P_\infty) \longrightarrow 0 \\ & & \searrow \theta^\wedge & & \downarrow \pi_\infty^\wedge \\ & & & & \mathcal{D}(P[n], M) \end{array}$$

The map  $\psi^\wedge$  is surjective since the map

$$\varphi[1]^\wedge : \coprod_{q \geq 0} \mathcal{D}(P[n], P_q[1]) \rightarrow \coprod_{q \geq 0} \mathcal{D}(P[n], P_q[1])$$

is injective. If we prove that the kernel of  $\theta^\wedge$  is contained in the image of  $\varphi^\wedge$ , then we would have the injectivity of  $\pi_\infty^\wedge$  by an easy diagram chase. Let

$$g = \begin{bmatrix} g_0 & g_1 & \dots & g_s & 0 & \dots \end{bmatrix}^t : P[n] \rightarrow \coprod_{q \geq 0} P_q$$

be an element of the kernel of  $\theta^\wedge$ . Then

$$\pi_0 g_0 + \dots + \pi_s g_s = 0$$

implies

$$\pi_s(f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s) = 0$$

and so the morphism

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s$$

factors through  $\alpha_s$ :

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s = \alpha_s \gamma_s : P[n] \rightarrow C_s \rightarrow P_s.$$

By construction of  $Z_s$  we have that  $\gamma_s$  factors through  $\beta_s$ , and so

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s = \alpha_s \beta_s \xi_s.$$

This implies

$$f_s \dots f_0 g_0 + f_s \dots f_1 g_1 + \dots + f_s g_s = f_s \alpha_s \beta_s \xi_s = 0,$$

since  $f_s \alpha_s \beta_s = 0$  by construction of  $f_s$ . Therefore, the morphism

$$h : P[n] \rightarrow \coprod_{q \geq 0} P_q$$

with non-vanishing components

$$P[n] \rightarrow P_r \rightarrow \coprod_{q \geq 0} P_q$$

induced by

$$g_r + \dots + f_{r-1} \dots f_1 g_1 + f_{r-1} \dots f_0 g_0 : P[n] \rightarrow P_r$$

with  $0 \leq r \leq s$ , satisfies  $\varphi^\wedge(h) = g$ . ✓

**Corollary 4.** *Let  $\mathcal{A}$  be a dg category and let  $\mathcal{P}$  be a set of objects of  $\mathcal{D}^- \mathcal{A}$  such that:*

- 1) *it is both right bounded and dually right bounded,*
- 2) *its objects are compact in  $\text{Tri}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$ ,*
- 3)  *$(\mathcal{DA})(P, P'[i]) = 0$  for each  $P, P' \in \mathcal{P}$  and  $i \geq 1$ .*

Then  $\text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$  is exhaustively generated to the left by  $\mathcal{P}$ .

*Proof.* Put  $\mathcal{D} := \text{Tria}(\mathcal{P}) \cap \mathcal{D}^- \mathcal{A}$ . Since  $\mathcal{P}$  is right bounded, then  $\mathcal{P}$  is contained in  $\mathcal{D}$  and for each integer  $k$  small coproducts of finite extensions of  $\text{Sum}(\mathcal{P}^+[k])$  are in  $\mathcal{D}$ . Also, Proposition 4 guarantees that for each  $M \in \mathcal{D}$  there exists an integer  $k_M$  such that  $\mathcal{D}(P[n], M) = 0$  for each  $P \in \mathcal{P}$  and  $n < k_M$ . Therefore, we can apply Proposition 6.  $\checkmark$

**Corollary 5.** *Let  $\mathcal{A}$  be a dg category. The following assertions are equivalent:*

- 1)  $\mathcal{D}^- \mathcal{A}$  is a recollement of  $\mathcal{D}^- \mathcal{B}$  and  $\mathcal{D}^- \mathcal{C}$ , for certain dg categories  $\mathcal{B}$  and  $\mathcal{C}$  with cohomology concentrated in non-positive degrees.
- 2) There exist sets  $\mathcal{P}$ ,  $\mathcal{Q}$  in  $\mathcal{D}^- \mathcal{A}$  such that:
  - 2.1)  $\mathcal{P}$  and  $\mathcal{Q}$  are right bounded.
  - 2.2)  $\mathcal{P}$  and  $\mathcal{Q}$  are dually right bounded.
  - 2.3) The objects of  $\mathcal{P}$  are compact in  $\mathcal{DA}$  and satisfy

$$(\mathcal{DA})(P, P'[i]) = 0$$

for all  $P, P' \in \mathcal{P}$  and  $i \geq 1$ .

- 2.4) The objects of  $\mathcal{Q}$  are compact in  $\text{Tria}(\mathcal{Q}) \cap \mathcal{D}^- \mathcal{A}$  and satisfy

$$(\mathcal{DA})(Q, Q'[i]) = 0$$

for all  $Q, Q' \in \mathcal{Q}$  and  $i \geq 1$ .

- 2.5)  $(\mathcal{DA})(P[i], Q) = 0$  for each  $P \in \mathcal{P}$ ,  $Q \in \mathcal{Q}$  and  $i \in \mathbf{Z}$ .
- 2.6)  $\mathcal{P} \cup \mathcal{Q}$  generates  $\mathcal{DA}$ .

*Proof.* 1)  $\Rightarrow$  2) Is similar to the corresponding implication in Theorem 1. The fact that the dg categories  $\mathcal{B}$  and  $\mathcal{C}$  have cohomology concentrated in non-positive degrees is reflected in the fact that

$$(\mathcal{DA})(P, P'[i]) = (\mathcal{DA})(Q, Q'[i]) = 0$$

for each  $P, P' \in \mathcal{P}$ ,  $Q, Q' \in \mathcal{Q}$  and  $i \geq 1$ .

2)  $\Rightarrow$  1) Thanks to Corollary 4, conditions 2.3 and 2.4 of Theorem 1 are satisfied. Therefore, that Proposition (and its proof) ensures that  $\mathcal{D}^- \mathcal{A}$  is a recollement of  $\mathcal{D}^- \mathcal{B}$  and  $\mathcal{D}^- \mathcal{C}$ , where  $\mathcal{B}$  is a fibrant replacement of  $\mathcal{Q}$  and  $\mathcal{C}$  is a fibrant replacement of  $\mathcal{P}$ . Finally, the fact that

$$(\mathcal{DA})(P, P'[i]) = (\mathcal{DA})(Q, Q'[i]) = 0$$

for each  $P, P' \in \mathcal{P}$ ,  $Q, Q' \in \mathcal{Q}$  and  $i \geq 1$  implies that  $\mathcal{B}$  and  $\mathcal{C}$  have cohomology concentrated in non-positive degrees.  $\checkmark$

### 5.3. Recollement of right bounded derived categories of algebras.

**Definition 5.** Let  $A$  be an ordinary algebra. If  $M$  is a complex of  $A$ -modules, the *graded support* of  $M$  is the set of integers  $i \in \mathbf{Z}$  such that  $M^i \neq 0$ . In case  $M$  is a bounded complex, we consider

$$w(M) := \sup\{i \in \mathbf{Z} \mid M^i \neq 0\} - \inf\{i \in \mathbf{Z} \mid M^i \neq 0\} + 1$$

and call it the *width* of  $M$ . Suppose now that  $P$  is a bounded complex of projective  $A$ -modules, so that  $P$  is a dually right bounded object of  $\mathcal{D}^- A$  (cf. Proposition 4), and  $M \in \mathcal{D}^- A$  is any object of the right bounded derived category. Unless  $M \in \text{Tria}_{\mathcal{DA}}(P)^\perp$ , there is a well-defined integer  $k_M := \inf\{n \in \mathbf{Z} \mid (\mathcal{DA})(P[n], M) \neq 0\}$ .

**Lemma 4.** *Let  $A$  be an ordinary algebra. Let  $P$  be a bounded complex of projective  $A$ -modules such that  $(\mathcal{D}A)(P, P[i]) = 0$ , for all  $i > 0$ , and the canonical morphism  $(\mathcal{D}A)(P, P[i])^{(\Lambda)} \rightarrow (\mathcal{D}A)(P, P[i]^{(\Lambda)})$  is an isomorphism, for every integer  $i$  and every set  $\Lambda$ . Let  $M$  be an object of  $\text{Tri}_{\mathcal{D}A}(P) \cap \mathcal{D}^-A$ . There exists a sequence of inflations  $0 = P_{-1} \rightarrow P_0 \rightarrow P_1 \rightarrow \dots$  in  $\mathcal{C}A$ , whose colimit is denoted by  $P_\infty$ , satisfying the following properties:*

- 1)  $P_\infty$  is isomorphic to  $M$  in  $\mathcal{D}A$ .
- 2)  $P_n/P_{n-1}$  belongs to  $\text{Sum}(\{P\}^+[k_M + n])$ , for each  $n \geq 0$ .
- 3) If  $n \geq w(P) - k_M$  the graded supports of  $P$  and  $P_\infty/P_n$  are disjoint.

*Proof.* Imitating the proof of Proposition 6, we shall construct a filtration satisfying conditions 2) and 3), leaving for the last moment the verification of condition 1). *First step: condition 2).* Note that in the proof of that proposition, we start with  $P_0 \in \text{Sum}(P[i] : i \geq k_M)$  and then, at each step,  $P_{q+1}$  appears in a triangle

$$Z_q \xrightarrow{\alpha_q \beta_q} P_q \xrightarrow{f_q} P_{q+1} \rightarrow Z_q[1],$$

where  $Z_q$  is a coproduct of shifts  $P[i]$ , with  $i \geq k_M - 1$ . Working in  $\mathcal{C}A$  and bearing in mind that  $Z_q$  is cofibrant (it is a right bounded complex of projective  $A$ -modules), we can assume without loss of generality that  $f_q$  is the mapping cone of a cochain map  $Z_q \rightarrow P_q$  and, as a consequence, that  $f_q$  is an inflation in  $\mathcal{C}A$  appearing in a conflation

$$P_q \xrightarrow{f_q} P_{q+1} \rightarrow Z_q[1],$$

where  $Z_q[1]$  is a coproduct in  $\mathcal{C}A$  of shifts  $P[i]$ ,  $i \geq k_M$ . We shall prove by induction on  $q \geq 0$  that one can choose  $Z_q[1] \in \text{Sum}(\{P\}^+[q + 1 + k_M])$  or, equivalently, that  $Z_q \in \text{Sum}(\{P\}^+[q + k_M])$ . Since  $Z_q$  is defined via a map  $\beta_q : Z_q \rightarrow C_q$  such that

$$\beta_q^\wedge : (\mathcal{D}A)(P[i], Z_q) \rightarrow (\mathcal{D}A)(P[i], C_q)$$

is surjective for all  $i \in \mathbf{Z}$ , our task reduces to prove that  $(\mathcal{D}A)(P[i], C_q) \neq 0$  implies  $i \geq q + k_M$ . We leave as an exercise checking that for  $q = 0$ . Provided it is true for  $q - 1$ , we apply the homological functor  $(\mathcal{D}A)(P[i], ?)$  to the triangle

$$Z_{q-1} \xrightarrow{\beta_{q-1}} C_{q-1} \xrightarrow{u_{q-1}} C_q \rightarrow Z_{q-1}[1]$$

and, bearing in mind that  $(\mathcal{D}A)(P[i], Z_{q-1}) \rightarrow (\mathcal{D}A)(P[i], C_{q-1})$  is surjective, we get that  $(\mathcal{D}A)(P[i], C_q) \rightarrow (\mathcal{D}A)(P[i], Z_{q-1}[1])$  is injective. As a consequence, the inequality  $(\mathcal{D}A)(P[i], C_q) \neq 0$  implies that  $(\mathcal{D}A)(P[i], Z_{q-1}[1]) \neq 0$  and the induction hypothesis guarantees that  $Z_{q-1}$  is a coproduct of shifts  $P[j]$ , with  $j \geq q - 1 + k_M$ . Then  $(\mathcal{D}A)(P[i], C_q) \neq 0$  implies that  $0 \neq (\mathcal{D}A)(P[i], P[j + 1]) = (\mathcal{D}A)(P, P[j + 1 - i])$ , for some  $j \geq q - 1 + k_M$ . Then  $i \geq q + k_M$  as desired. In conclusion, we can view the map  $f_q : P_q \rightarrow P_{q+1}$  as an inflation in  $\mathcal{C}A$  whose cokernel is isomorphic in  $\mathcal{C}A$  to a coproduct of shifts  $P[i]$ , with  $i \geq q + 1 + k_M$ .

*Second step: condition 3).* If now  $n \geq 0$  is any natural number, then  $P_\infty/P_n$  admits a filtration

$$0 = P_n/P_n \rightarrow P_{n+1}/P_n \rightarrow \dots$$

in  $\mathcal{C}A$ , where the quotient of two consecutive factors is a coproduct of shifts  $P[i]$ , with  $i \geq n + k_M$ . If  $n \geq w(P) - k_M$ , then any such index  $i$  satisfies  $i \geq w(P)$  and then the graded supports of  $P$  and  $P[i]$  are disjoint. As a result the graded supports of  $P$  and  $P_\infty/P_n$  are disjoint whenever  $n \geq w(P) - k_M$ .

*Third step: condition 1).* Finally, in order to prove condition 1), notice that the argument in the final part of the proof of Proposition 6 can be repeated, as soon as we are able to prove that the canonical morphism  $\coprod_{n \geq 0} (\mathcal{D}A)(P[i], P_n) \rightarrow (\mathcal{D}A)(P[i], \coprod_{n \geq 0} P_n)$  is an isomorphism, for every integer  $i \in \mathbf{Z}$ . It is not difficult to reduce that to the case in which  $i = 0$ . For that we fix  $n \geq w(P) - k_M$  large enough so that also the graded supports of  $P[1]$  and  $P_\infty/P_n$  are disjoint. Then we get a conflation in  $\mathcal{C}A$

$$\left( \coprod_{k \leq n} P_k \right) \oplus \left( \coprod_{k > n} P_n \right) \rightarrow \coprod_{k \geq 0} P_k \rightarrow \coprod_{k > n} P_k/P_n.$$

That conflation of  $\mathcal{C}A$  gives rise to the corresponding triangle of  $\mathcal{D}A$ . But the right term in the above conflation has a graded support which is disjoint with those of  $P$  and  $P[1]$ . That implies that

$$(\mathcal{D}A)(P, \coprod_{k > n} P_k/P_n) = 0 = (\mathcal{D}A)(P, \coprod_{k > n} P_k/P_n[-1])$$

and also

$$\coprod_{k > n} (\mathcal{D}A)(P, P_k/P_n) = 0 = \coprod_{k > n} (\mathcal{D}A)(P, P_k/P_n[-1]).$$

We then get a commutative diagram with horizontal isomorphisms:

$$\begin{array}{ccc} \left( \coprod_{k \leq n} (\mathcal{D}A)(P, P_k) \right) \oplus \left( \coprod_{k > n} (\mathcal{D}A)(P, P_n) \right) & \xrightarrow{\sim} & \coprod_{k \geq 0} (\mathcal{D}A)(P, P_k) \\ \downarrow \text{can} & & \downarrow \text{can} \\ \left( (\mathcal{D}A)(P, \coprod_{k \leq n} P_k) \right) \oplus \left( (\mathcal{D}A)(P, \coprod_{k > n} P_n) \right) & \xrightarrow{\sim} & (\mathcal{D}A)(P, \coprod_{k \geq 0} P_k) \end{array}$$

The proof will be finished if we are able to prove, for any fixed natural number  $n$ , that  $(\mathcal{D}A)(P[i], ?)$  preserves small coproducts of objects in  $\text{Sum}(\{P\}^+)^{*n}$  for every  $i \in \mathbf{Z}$ . Let us prove it. From the hypotheses on  $P$  and the fact if  $i > w(P)$  then  $(\mathcal{D}A)(P, P[i]^{(\Lambda)}) = 0$  for every set  $\Lambda$ , one readily sees that, for every integer  $m$  and every family of exponent sets  $(\Lambda_i)_{i \geq m}$ , the canonical morphism  $\coprod_{i \geq m} (\mathcal{D}A)(P, P[i]^{(\Lambda_i)}) \rightarrow (\mathcal{D}A)(P, \coprod_{i \geq m} P[i]^{(\Lambda_i)})$  is an isomorphism. Our goal is then attained for  $n = 0$  and an easy induction argument gets the job done for every  $n \geq 0$ .  $\checkmark$

**Definition 6.** An object  $M$  of a (typically compactly generated) triangulated category  $\mathcal{D}$  is *exceptional* if  $\mathcal{D}(M, M[i]) = 0$  for every integer  $i \neq 0$ .

Now we can deduce the following theorem:

**Theorem 2.** Let  $A$ ,  $B$  and  $C$  be ordinary algebras. The following assertions are equivalent:

- 1)  $\mathcal{D}^-A$  is a recollement of  $\mathcal{D}^-C$  and  $\mathcal{D}^-B$ .
- 2) There are two objects  $P, Q \in \mathcal{D}^-A$  satisfying the following properties:
  - 2.1) There are isomorphisms of algebras  $C \cong (\mathcal{D}A)(P, P)$  and  $B \cong (\mathcal{D}A)(Q, Q)$ .
  - 2.2)  $P$  is exceptional and isomorphic in  $\mathcal{D}A$  to a bounded complex of finitely generated projective  $A$ -modules.

- 2.3) For every set  $\Lambda$  and every non-zero integer  $i$  we have  $(\mathcal{D}A)(Q, Q^{(\Lambda)}[i]) = 0$ , the canonical map  $(\mathcal{D}A)(Q, Q)^{(\Lambda)} \rightarrow (\mathcal{D}A)(Q, Q^{(\Lambda)})$  is an isomorphism, and  $Q$  is isomorphic in  $\mathcal{D}A$  to a bounded complex of projective  $A$ -modules.
- 2.4)  $(\mathcal{D}A)(P, Q[i]) = 0$  for all  $i \in \mathbf{Z}$ .
- 2.5)  $P \oplus Q$  generates  $\mathcal{D}A$ .

*Proof.* 1)  $\Rightarrow$  2) is a particular case of the proof of the corresponding implication in Corollary 5, where we take into account Corollary 3 and the additional consideration that the dg categories are in this case ordinary algebras, whence having cohomology concentrated in degree zero. 2)  $\Rightarrow$  1) Taking  $\mathcal{P} = \{P\}$  and  $\mathcal{Q} = \{Q\}$ , one readily sees that these one-point sets satisfy conditions 2.1, 2.2, 2.3, 2.5 and 2.6 of Corollary 5. As for condition 2.4 it only remains to prove that  $Q$  is compact in  $\text{Tria}_{\mathcal{D}A}(Q) \cap \mathcal{D}^-A$ . For this, let  $(M_j)_{j \in J}$  be a family of objects in  $\text{Tria}(Q) \cap \mathcal{D}^-A$  having a coproduct, say  $M$ , in that subcategory and denote by  $q_j : M_j \rightarrow M$  the injections. Of course, we have that  $\sup\{i \in \mathbf{Z} \mid H^i(M_j) \neq 0\} \leq \sup\{i \in \mathbf{Z} \mid H^i(M) \neq 0\}$ , for every  $j \in J$ . Then the coproduct  $\coprod_{j \in J} M_j$  of the family in  $\mathcal{D}A$  belongs to  $\mathcal{D}^-A$  and thus to  $\text{Tria}(Q) \cap \mathcal{D}^-A$ . This easily implies that  $M \cong \coprod_{j \in J} M_j$  and the injection  $q_j : M_j \rightarrow M$  gets identified with the canonical injection  $M_j \rightarrow \coprod_{k \in J} M_k$ . For each  $j \in J$  we consider the complex  $Q_{j,\infty}$  and the filtration

$$0 = Q_{j,-1} \rightarrow Q_{j,0} \rightarrow Q_{j,1} \rightarrow \dots$$

given by Lemma 4 for  $M_j$ , where we have replaced the letter “P” by the letter “Q” to avoid confusion with the object  $P$ . Notice that  $k_M \leq k_{M_j}$  for every  $j \in J$ . Therefore, the integer  $r := \inf\{k_{M_j}\}_{j \in J}$  is well defined. If we fix  $n \in \mathbf{N}$  such that  $n + r > w(Q)$ , then  $n > w(Q) - k_{M_j}$ . Notice that [10, Lemma 5.3] implies that a countable composition of inflations of  $\mathcal{C}A$  is again an inflation of  $\mathcal{C}A$ . Then, for every  $j \in J$  we get a conflation in  $\mathcal{C}A$ ,

$$Q_{j,n} \rightarrow Q_{j,\infty} \rightarrow Q_{j,\infty}/Q_{j,n},$$

By Lemma 4, the right term of this conflation has a graded support which is disjoint with that of  $Q$  and  $Q[1]$  (enlarging  $n$  if necessary). Then we get a commutative diagram:

$$\begin{array}{ccccc} \coprod_J (\mathcal{D}A)(Q, Q_{j,n}) & \xrightarrow{\sim} & \coprod_J (\mathcal{D}A)(Q, Q_{j,\infty}) & \xrightarrow{\sim} & \coprod_J (\mathcal{D}A)(Q, M_j) \\ \downarrow \text{can} & & \downarrow \text{can} & & \downarrow \text{can} \\ (\mathcal{D}A)(Q, \coprod_J Q_{j,n}) & \xrightarrow{\sim} & (\mathcal{D}A)(Q, \coprod_J Q_{j,\infty}) & \xrightarrow{\sim} & (\mathcal{D}A)(Q, \coprod_J M_j) \end{array}$$

The fact that the leftmost vertical map is a bijection has been proved in the third step of the proof of Lemma 4, and so we are done.  $\checkmark$

## 6. MORE THAN AN EXCEPTIONAL OBJECT

**6.1. The mismatch.** Theorem 2 is very close to the following theorem of S. König [16, Theorem 1]:

**Theorem.** *Let  $A$ ,  $B$  and  $C$  be ordinary algebras. The following assertions are equivalent:*

- 1)  $\mathcal{D}^-A$  is a recollement of  $\mathcal{D}^-C$  and  $\mathcal{D}^-B$ .
- 2) There are two objects  $P, Q \in \mathcal{D}^-A$  satisfying the following properties:
  - 2.1) There are isomorphisms of algebras  $C \cong (\mathcal{D}A)(P, P)$  and  $B \cong (\mathcal{D}A)(Q, Q)$ .

- 2.2)  $P$  is exceptional and isomorphic in  $\mathcal{DA}$  to a bounded complex of finitely generated projective  $A$ -modules.
- 2.3)  $Q$  is exceptional, it is isomorphic in  $\mathcal{DA}$  to a bounded complex of projective  $A$ -modules and the functor  $\mathrm{Hom}_A(Q, ?) : \mathrm{Mod} A \rightarrow \mathrm{Mod} k$  preserves small coproducts of copies of  $Q$ .
- 2.4)  $(\mathcal{DA})(P, Q[i]) = 0$  for all  $i \in \mathbf{Z}$ .
- 2.5)  $P \oplus Q$  generates  $\mathcal{DA}$ .

The reader will have noticed that we changed S. König's condition that  $Q$  is exceptional for the stronger condition that  $(\mathcal{DA})(Q, Q[i]^{(\Lambda)}) = 0$ , for all  $i \neq 0$  and all sets  $\Lambda$ . In what follows we will show that this stronger condition is needed in order for the theorem to be valid.

**6.2. Some results on countable von Neumann regular algebras.** We thank J. Trlifaj for giving us an example [24, Lemma 6.3.14 and Example 6.3.15] that was at the basis for the following development.

**Lemma 5.** *If  $A$  is a countable von Neumann regular algebra, then it is hereditary on both sides.*

*Proof.* Since  $A$  is countable its pure global dimension on either side is smaller or equal than 1 [8, Théorem 7.10] and, since  $A$  is von Neumann regular, we conclude that  $A$  is hereditary on both sides [8, Proposition 10.3].  $\checkmark$

**Lemma 6.** *Let  $A$  be a countable simple von Neumann regular algebra which is not semisimple. If  $Q$  is an injective  $A$ -module then the functor  $\mathrm{Hom}_A(Q, ?) : \mathrm{Mod} A \rightarrow \mathrm{Mod} k$  preserves small coproducts.*

*Proof.* *First step: the countable sequence of submodules.* of submodules of  $Q$  where  $Q_n := f^{-1}(\coprod_{i=0}^n M_i)$ . Notice that  $Q = \bigcup_{n \in \mathbf{N}} Q_n$  and that for every  $n \in \mathbf{N}$  we have  $Q_n \neq Q$ . This implies that we can choose a sequence  $n_0 < n_1 < \dots$  of natural numbers such that  $Q_i$  is strictly contained in  $Q_{i+1}$  whenever  $i = n_t$  for some  $t \in \mathbf{N}$ . *Second step: the countable sequence of idempotents.* Let  $e_t$ ,  $t \in \mathbf{N}$ , be a sequence of mutually orthogonal non-zero idempotents of  $A$ . Since  $Ae_tA$  is a non-zero two-sided ideal of the simple algebra  $A$ , then  $Ae_tA = A$  and so  $Q_{n_t} = Q_{n_t}A = Q_{n_t}e_tA$ . Therefore, for each  $t \in \mathbf{N}$  there exists an element  $x_t \in Q_{n_t}e_t$  which does not belong to  $Q_{n_{t-1}}$ .

$$g : \bigoplus_{t \in \mathbf{N}} e_tA \rightarrow Q, \quad \sum_{t \in \mathbf{N}} a_t \mapsto \sum_{t \in \mathbf{N}} x_t a_t.$$

Since  $Q$  is injective,  $g$  extends to  $A$  and so there exists an element  $x \in Q$  such that for every  $t \in \mathbf{N}$  we have that  $g(e_t) = x_t e_t = x_t = x e_t$ . If  $s$  is a natural number such that  $x \in Q_{n_s}$ , then  $x_t \in Q_{n_s}$  for every  $t \in \mathbf{N}$ , which contradicts the choice of the elements  $x_t$ .  $\checkmark$

**Lemma 7.** *Let  $A$  be a countable simple von Neumann regular algebra which is not right Noetherian. If  $Q$  is an injective cogenerator of  $\mathrm{Mod} A$  containing an isomorphic copy of every cyclic module, then  $\mathrm{Ext}_A^1(Q, Q^{(\mathbf{N})}) \neq 0$ .*

*Proof.* *First step:  $Q^{(\mathbf{N})}$  is not injective.* *Second step:  $\mathrm{Ext}_A^1(Q, Q^{(\mathbf{N})}) \neq 0$ .* Since  $Q^{(\mathbf{N})}$  is not injective, Baer's criterion implies that [1, Theorem 18.3] there exists a cyclic  $A$ -module  $M$  such that  $\mathrm{Ext}_A^1(M, Q^{(\mathbf{N})}) \neq 0$ . We fix a monomorphism



$j : M \longrightarrow Q$ , which we view as an inclusion. Now, by applying  $\text{Hom}_A(Q, ?)$  and  $\text{Hom}_A(M, ?)$  to the minimal injective coresolution

$$0 \rightarrow Q^{(\mathbb{N})} \rightarrow E(Q^{(\mathbb{N})}) \rightarrow E' \rightarrow 0$$

we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} \text{Hom}_A(Q, E') & \longrightarrow & \text{Ext}_A^1(Q, Q^{(\mathbb{N})}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \text{Hom}_A(M, E') & \longrightarrow & \text{Ext}_A^1(M, Q^{(\mathbb{N})}) & \longrightarrow & 0 \end{array}$$

where the left vertical arrow is the restriction map, and it is surjective because  $E'$  is injective. Then, the right vertical arrow is surjective, which implies that  $\text{Ext}_A^1(Q, Q^{(\mathbb{N})}) \neq 0$ .  $\checkmark$

**Lemma 8.** *Let  $A$  be a countable simple von Neumann regular algebra which is not right Noetherian, and let  $Q$  be an injective cogenerator of  $\text{Mod } A$  containing an isomorphic copy of every cyclic module. Then an  $A$ -module  $M$  is zero whenever*

$$\text{Hom}_A(Q, M) = \text{Ext}_A^1(Q, M) = 0.$$

*Proof.* Consider a minimal injective coresolution  $0 \rightarrow M \rightarrow E(M) \rightarrow E'(M) \rightarrow 0$  of an  $A$ -module  $M$  such that  $\text{Hom}_A(Q, M) = \text{Ext}_A^1(Q, M) = 0$ . *First step:* If  $M \neq 0$  then  $E'(M) \neq 0$ . Indeed, if  $E'(M) = 0$  then  $M$  is injective and so it contains the injective envelope of any non-zero cyclic submodule of  $M$ , which would be a non-zero direct summand of  $Q$  and  $M$ . This implies  $\text{Hom}_A(Q, M) \neq 0$ , which is a contradiction.

*Second step:*  $M = 0$ . Suppose not and let  $C$  be a non-zero cyclic submodule of  $E'(M)$ , so that its injective envelope  $Q' := E(C)$  is a direct summand of  $E'(M)$ . Fix a section  $v : Q' \rightarrow E'(M)$ . Since  $\text{Ext}_A^1(Q', M) = 0$ , there exists a morphism of  $A$ -modules  $f : Q' \rightarrow E(M)$  which fits in the following commutative diagram

$$\begin{array}{ccccccc} & & & Q' & & & \\ & & & \downarrow v & & & \\ & & f & \swarrow & & & \\ 0 & \longrightarrow & M & \longrightarrow & E(M) & \xrightarrow{p} & E'(M) \longrightarrow 0 \end{array}$$

Then  $f$  is a monomorphism and  $f(Q')$  is a direct summand of  $E(M)$  isomorphic to  $Q'$  and such that  $p$  induces an isomorphism  $\pi : f(Q') \rightarrow v(Q')$ . Hence, we can rewrite the short exact sequence above as

$$0 \rightarrow M \rightarrow E \oplus f(Q') \xrightarrow{\begin{bmatrix} \alpha & 0 \\ \beta & \pi \end{bmatrix}} E' \oplus Q' \longrightarrow 0$$

Notice that in this short exact sequence the kernel of the epimorphism, which is isomorphic to  $M$ , intersects in 0 with  $0 \oplus f(Q')$ . This implies that  $M$  is not essential in  $E(M)$ , which is absurd.  $\checkmark$

**Example 4.** Any countable direct limit of countable simple Artinian algebras is a countable simple von Neumann regular algebra which is not right Noetherian. A typical case is given as follows. Consider the direct limit  $\varinjlim \mathcal{M}_{2^n \times 2^n}(\mathbb{K})$ , where

$\mathbb{K}$  is a countable field and the ring morphism  $\mathcal{M}_{2^n \times 2^n}(\mathbb{K}) \rightarrow \mathcal{M}_{2^{n+1} \times 2^{n+1}}(\mathbb{K})$  maps the matrix  $U$  onto the matrix given by the block decomposition  $\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$ .

**6.3. A counterexample.** Let  $H$  be a countable simple von Neumann regular algebra which is not right Noetherian. Let  $\mathcal{L}(H)$  be the family of right ideals of  $H$  and let  $Q'$  be the injective envelope of  $\coprod_{I \in \mathcal{L}(H)} H/I$ .

**Remark 3.** Notice that  $\text{End}_H(Q')$  is not countable for there exist two obvious injective maps

$$\prod_{I \in \mathcal{L}(H)} \text{End}_H(H/I) \rightarrow \text{End}_H\left(\prod_{I \in \mathcal{L}(H)} H/I\right) \rightarrow \text{End}_H(Q').$$

Let  $C$  be any unital subalgebra of  $\text{End}_H(Q')$ . Take  $A$  to be the triangular matrix algebra

$$A := \begin{bmatrix} C & Q' \\ 0 & H \end{bmatrix}.$$

The category  $\text{Mod } A$  admits a nice description in terms of  $\text{Mod } C$  and  $\text{Mod } H$  (cf. [2, subsection III.2]) that we will use without explicit mention. Consider the idempotent

$$e := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since  $AeA = eA$ , it turns out that the idempotent ideal  $I := AeA$  is a projective right  $A$ -module and so the canonical projection

$$\pi : A \rightarrow A/I$$

is a homological epimorphism (cf. [24, Example 5.3.4]). Of course, we have isomorphisms of algebras  $A/I \cong H$  and  $\text{End}_H(I) \cong eAe \cong C$ . Therefore,  $\mathcal{D}A$  is a recollement of  $\mathcal{D}H$  and  $\mathcal{D}C$  as follows:

$$\mathcal{D}H \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}A \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}C.$$

It induces a TTF triple  $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$  on  $\mathcal{D}A$  where  $\mathcal{X} = \text{Tri}_{\mathcal{D}A}(I)$  and  $\mathcal{Y}$  consists of those complexes isomorphic in  $\mathcal{D}A$  to complexes of  $H$ -modules regarded as  $A$ -modules. Moreover, Example 3 tells us that this TTF triple restricts to a TTF triple on  $\mathcal{D}^-A$  which expresses  $\mathcal{D}^-A$  as a recollement of  $\mathcal{D}^-H$  and  $\mathcal{D}^-C$ :

$$\mathcal{D}^-H \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^-A \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^-C.$$

In particular,  $\mathcal{D}^-H$  is the triangle quotient of  $\mathcal{D}^-A$  by  $\mathcal{D}^-C$ .

We claim that  $P = eA$  and  $Q = [0, Q'; 0]$  satisfy all the conditions of S. König's theorem (see subsection 6.1): *Condition 2.2:*  $P$  is clearly an exceptional object of  $\mathcal{D}A$  since it is a projective  $A$ -module.

*Condition 2.3:*

- Let us check that  $Q$  is exceptional. Since the canonical functor  $\mathcal{D}H \rightarrow \mathcal{D}A$  is fully faithful, we just have to check that  $Q'$  is an exceptional object of  $\mathcal{D}H$ , which is true because  $Q'$  is an injective  $H$ -module.

- Since  $H$  is hereditary,  $Q'$  admits a projective resolution of length 1. But the canonical functor  $\text{Mod } H \rightarrow \text{Mod } A$  preserves projective objects, and thus  $Q$  admits a projective resolution of length 1. This shows that  $Q$  is isomorphic in  $\mathcal{D}A$  to a bounded complex of projective  $A$ -modules.
- To check that  $\text{Hom}_A(Q, ?)$  preserves small coproducts of copies of  $Q$  one uses the fact that  $\mathcal{D}H \rightarrow \mathcal{D}A$  is fully faithful and applies Lemma 6 with  $Q'_H$ .

*Condition 2.4:* Since  $P$  is a projective  $A$ -module, we only have to check that  $(\mathcal{D}A)(P, Q) = 0$ , but this is clear since

$$(\mathcal{D}A)(P, Q) \cong \text{Hom}_A(P, Q) \cong Qe = 0.$$

*Condition 2.5:* Let  $M$  be a complex of  $A$ -modules such that  $(\mathcal{D}A)(P[i], M) = (\mathcal{D}A)(Q[i], M) = 0$  for each integer  $i$ . Consider the triangle

$$\tau_{\mathcal{X}}M \rightarrow M \rightarrow \tau_{\mathcal{Y}}M \rightarrow (\tau_{\mathcal{X}}M)[1]$$

of  $\mathcal{D}A$ . Since  $(\mathcal{D}A)(P, \tau_{\mathcal{Y}}M[i]) = 0$  for each integer  $i$ , then  $(\mathcal{D}A)(P, \tau_{\mathcal{X}}M[i]) = 0$  for each integer  $i$ . Now, the fact that  $P$  generates  $\mathcal{X}$  implies  $\tau_{\mathcal{X}}M = 0$ , that is to say,  $M$  belongs to  $\mathcal{Y}$ . Therefore, we can assume that  $M$  is the image of a complex  $M'$  of  $H$ -modules by the canonical functor  $\mathcal{D}H \rightarrow \mathcal{D}A$ . Then, since  $H$  is right hereditary, for each integer  $i$  we have

$$\begin{aligned} 0 &= (\mathcal{D}A)(Q[i], M) \cong (\mathcal{D}H)(Q'[i], M') \cong (\mathcal{D}H)(Q'[i], \prod_{n \in \mathbf{Z}} H^n(M')[-n]) \cong \\ &\cong \prod_{n \in \mathbf{Z}} (\mathcal{D}H)(Q'[i], H^n(M')[-n]) \cong \prod_{n \in \mathbf{Z}} \text{Ext}_H^{-n-i}(Q', H^n(M')) \end{aligned}$$

Finally, Lemma 8 (applied with  $Q'_H$ ) tells us that  $M'$  is acyclic, that is to say,  $M = 0$  in  $\mathcal{D}A$ .

According to S. König's theorem,  $\mathcal{D}^-A$  is a recollement as follows:

$$\mathcal{D}^-(\text{End}_H(Q')) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^-A \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^-C.$$

In particular,  $\mathcal{D}^-(\text{End}_H(Q'))$  is the triangle quotient of  $\mathcal{D}^-A$  by  $\mathcal{D}^-C$ . Therefore,  $\mathcal{D}^-(\text{End}_H(Q'))$  is triangle equivalent to  $\mathcal{D}^-H$ . Let us fix a triangle equivalence  $F : \mathcal{D}^-(\text{End}_H(Q')) \xrightarrow{\sim} \mathcal{D}^-H$  and let us put  $F(\text{End}_H(Q')) =: T$ . Since  $H$  is right hereditary and  $T$  is a compact object of  $\mathcal{D}H$ , we deduce that  $T$  is isomorphic in  $\mathcal{D}H$  to a finite coproduct of stalk complexes  $M_i[n_i]$ ,  $1 \leq i \leq r$ ,  $n_i \in \mathbf{Z}$ , for some  $H$ -modules  $M_i$ . This implies that each  $M_i$  is compact in  $\mathcal{D}H$ . Therefore, each  $M_i$  is finitely presented and so (cf. [28, Proposition I.12.1, Corollary I.11.5]) it is a finitely generated projective  $H$ -module. Assume that  $r > 1$ , and, without loss of generality, that  $M_i \neq 0$  for each  $1 \leq i \leq r$  and that  $n_i \neq n_j$  for two different indexes  $i$  and  $j$ . Since  $T$  is exceptional, there exists an isomorphism of algebras  $\text{End}_{\mathcal{D}H}(T) \cong \bigoplus_{i=1}^r \text{End}_{\mathcal{D}H}(M_i)$  inducing a triangle equivalence  $\mathcal{D}H \simeq \bigoplus_{i=1}^r \mathcal{D}(\text{End}_{\mathcal{D}H}(M_i))$ . This implies (cf. [24, Example 1.7.15]) that there exists a non-zero central idempotent  $e$  of  $H$  different from 1, which contradicts the fact that  $H$  is a simple ring. projective  $H$ -module. Of course,  $T$  generates the triangulated category  $\mathcal{D}^-H$  and so it is also a generator of the abelian category  $\text{Mod } H$ . We have deduced that  $T$  is a finitely generated projective generator of  $\text{Mod } H$  and so  $\text{End}_H(T)$  is Morita equivalent to  $H$ . In particular, since  $\text{End}_H(Q') \cong \text{End}_{\mathcal{D}H}(T) \cong \text{End}_H(T)$ , we have that  $\text{End}_H(Q')$

is Morita equivalent to  $H$ . By the explicit description of Morita equivalences, this is impossible because  $H$  is countable and  $\text{End}_H(Q')$  is not.

**Remark 4.** S. König has pointed out to us that the construction of the functor  $F_{\text{in}}$  [26, Theorem 2.12] still yields a full embedding if  $T$  is a bounded complex of (not necessarily finitely generated) projective  $A$ -modules such that  $(\mathcal{D}A)(T, T^{(S)}[i]) = 0$  for every set  $S$  and every non-zero integer  $i$  and that, as a consequence, his proof of [16, Theorem 1] should still work assuming our hypothesis 2.3) of Theorem 2.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, APTDO. 4021, 30100, ESPINARDO, MURCIA, ESPAÑA

*E-mail address:* `pedronz@um.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, APTDO. 4021, 30100, ESPINARDO, MURCIA, ESPAÑA

*E-mail address:* `msaorinc@um.es`